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Cleavability over ordinals

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ABSTRACT

In this paper we show that if X is an infinite compactum cleavable over an ordinal, then X must be homeomorphic to an ordinal. X must also therefore be a LOTS. This answers two fundamental questions in the area of cleavability. We also leave it as an open question whether cleavability of an infinite compactum X over an ordinal λ implies X is embeddable into λ.

0. Introduction

A space X is said to be *cleavable* (or splittable) over a space Y *along* $A \subseteq X$ if there exists a continuous $f : X \to Y$ such that $f(A) \cap f(X \setminus A) = \emptyset$. A space X is *cleavable over* Y if it is cleavable over Y along all $A \subseteq X$. The topic was introduced by A.V. Arkhangel'skiĭ and D.B. Shakhmatov in [1], but it was in [2] that two of the main questions were stated:

Question 1. Let X be an infinite compactum cleavable over a linearly ordered topological space (LOTS) Y. Is X homeomorphic to a subspace of Y?

Question 2. Let X be an infinite compactum cleavable over a LOTS. Is X a LOTS?

Results related to these questions can be found in, but are not limited to, the following papers: [2–6].

In this paper, we show that if X is an infinite compactum cleavable over an ordinal, then X must be homeomorphic to an ordinal. These results supplement those in [5], which concern cleavability over ordinals less than or equal to ω_1.

Providing positive answers to Questions 1 and 2 allows for the comprehension of a difficult space X by associating it with a more well-known and well-understood space: in this case an uncountable ordinal.

Further, several papers have been devoted to describing the necessary and sufficient conditions for the linear orderability of a space X. (See [7] and [8] for examples.) The results of this paper provide an alternative characterization by which we may show when an infinite compactum is linearly orderable.

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Additionally, another popular area of research is the characterization of those spaces that are homeomorphic to an ordinal (see [9–11] for examples). The results of this paper add another characterization of those spaces homeomorphic to an ordinal.

The novelty of these characterizations is that, whereas the results of [7–11] rely on topological properties of a space X, our new characterizations rely on finding an appropriate ordinal λ, and an appropriate subset of $\mathscr{C}(X,\lambda)$. This shifts the focus from a topological exercise, to a functional one.

This paper is written in four sections. In the first section, we provide introductory definitions, observations and lemmas. The most important of these is Theorem 1.18; in this theorem, we show that any compact X cleavable over an ordinal λ such that X "hereditarily has a spine" must be homeomorphic to an ordinal. (A definition for this property is provided in Definition 1.11.) In the second and third sections, we show that every compact X cleavable over an ordinal must hereditarily have a spine. In the fourth section, we provide an answer to Questions 1 and 2. The second and third sections of this paper are heavily technical. The main result of this paper, that X is homeomorphic to an ordinal Y, is stated and proven in Theorem 4.4.

1. Introductory proofs

In this section we provide introductory definitions, observations, and lemmas. The most important theorem of this section is Theorem 1.18, in which we show that every infinite compactum that "hereditarily has a spine" (see Definition 1.11), and is cleavable over an ordinal, must be homeomorphic to an ordinal. This provides the foundation for the rest of the paper, in which we prove that every infinite compactum cleavable over an ordinal must hereditarily have a spine.

We begin by stating several well-known definitions, observations, and lemmas. The first theorem is from [3].

Theorem 1.1. *If X is a compactum cleavable over a T_2-space, then X is T_2.*

Lemma 1.2. *If X is a compactum cleavable over a scattered space Y, then X must be scattered.*

Proof. Assume for a contradiction that X contains a dense-in-itself subset D, and consider $\overline{D}$. We know $\overline{D}$ is compact T_2, and perfect, therefore by [12] it is resolvable. Let $\overline{D} = A \cup B$, where $\overline{A} = \overline{B} = \overline{D}$, and $A \cap B = \emptyset$. No function from X to a scattered space can cleave apart A and B, thus X cannot contain a dense-in-itself subset. $\square$

Definition 1.3. For ordinal numbers α, the α-th *derived set* of a topological space X is defined by transfinite induction as follows:

- $X^0 = X$.
- $X^{\alpha+1} = (X^\alpha)'$.
- $X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha$ for limit ordinals λ.

The smallest ordinal α such that $X^{\alpha+1} = X^\alpha$ is called the *Cantor–Bendixson rank* of X, written as $\mathrm{CB}(X)$. Lastly, let $I_\beta(X) = X^\beta \setminus X^{\beta+1}$.

The following observations are well known, but may also be found in [13].

Observation 1.4. *For a scattered space X, the Cantor–Bendixson rank is the least ordinal μ such that X^μ is empty.*

Observation 1.5. *If X is a compact scattered topological space, then the Cantor–Bendixson rank of X must be a successor ordinal.*

It follows from these observations that if X is compact and scattered, and $\mathrm{CB}(X) = \beta + 1$, then X^β is the last non-empty derived set of X. By compactness, $|X^\beta| < \omega$; from this, we have the following definitions:

Definition 1.6. Let X be a compact scattered space. If $\mathrm{CB}(X) = \beta + 1$, we say X is *simple* if $|X^\beta| = 1$. We say X is *simple with* $\hat{x}$ if $\hat{x}$ is the only element of X^β.

Definition 1.7. Let X be a scattered topological space. For $x \in X$, we use $\mathrm{CB}^*(x)$ to be the greatest ordinal β such that $x \in X^\beta$.

Definition 1.8. Let X be a compact scattered space. If X^β is finite and contains exactly n-many points, the pair (β, n) is called the *characteristic* of X.