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Strongly complete almost maximal left invariant topologies on groups

Valentin Keyantuo [a], Yevhen Zelenyuk [b,*,1]

[a] Department of Mathematics, University of Puerto Rico, PO Box 70377, San Juan, PR 00936-8377, USA
[b] School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa

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ABSTRACT

Let G be a countably infinite group. A topology on G is left invariant if left translations are continuous. A left invariant topology is strongly complete if it is regular and for every partition $\{U_n\colon\ n<\omega\}$ of G into open sets, there is a neighborhood V of 1 such that for every $x\in G$, $\{n<\omega\colon\ (xV)\cap U_n\neq\emptyset\}$ is finite. We show that assuming MA, for every $n\in\mathbb{N}$, there is a strongly complete left invariant topology $\mathcal{T}$ on G with exactly n nonprincipal ultrafilters converging to 1, and in the case $G=\bigoplus_\omega\mathbb{Z}_2$, $\mathcal{T}$ can be chosen to be a group topology. We also show that it is consistent with ZFC that if G can be embedded algebraically into a compact group, then there are no such topologies on G.

1. Introduction

Throughout the paper, G will be an arbitrary countably infinite discrete group.

The operation of G extends to the Stone–Čech compactification βG of G so that for each $a\in G$, the left translation $\beta G\ni x\mapsto ax\in\beta G$ is continuous, and for each $q\in\beta G$, the right translation $\beta G\ni x\mapsto xq\in\beta G$ is continuous. We take the points of βG to be the ultrafilters on G, the principal ultrafilters being identified with the points of G, and $G^*=\beta G\setminus G$. The topology of βG is generated by taking as a base the subsets $\overline{A}=\{p\in\beta G\colon\ A\in p\}$ where $A\subseteq G$. For $p,q\in\beta G$, the ultrafilter pq has a base consisting of subsets of the form $\bigcup_{x\in A}xB_x$ where $A\in p$ and $B_x\in q$. See [1] for more information about βG.

A topology on G is *left invariant* if left translations are continuous. All topologies are assumed to satisfy the T_1 separation axiom. Every left invariant topology $\mathcal{T}$ on G determines a closed subsemigroup $\mathrm{Ult}(\mathcal{T})\subseteq\beta G$ consisting of all nonprincipal ultrafilters on G converging to 1 in $\mathcal{T}$ (see [9, Section 7.1]). It is called the *ultrafilter semigroup* of $\mathcal{T}$. A left invariant topology is *maximal* (*almost maximal*) if its ultrafilter semigroup is a singleton (finite). Not each closed subsemigroup of G^* is the ultrafilter semigroup of a left invariant topology. However, every finite subsemigroup is [7, Proposition 2.4]. If $S\subseteq G^*$ is a finite subsemigroup and $\mathcal{T}$ is the left invariant topology on G with $\mathrm{Ult}(\mathcal{T})=S$ (that is, the neighborhoods of 1 in $\mathcal{T}$ are the subsets of the form $\bigcup_{p\in S}A_p\cup\{1\}$ where $A_p\in p$ for each $p\in S$), then $\mathcal{T}$ is regular if and only if for every $p\in\beta G\setminus(S\cup\{1\})$, $(pS)\cap S=\emptyset$ [7, Proposition 2.12]. In particular, if $p\in G^*$ is an idempotent and $\mathcal{T}$ is the left invariant

* Corresponding author.
E-mail addresses: valentin.keyantuo1@upr.edu (V. Keyantuo), yevhen.zelenyuk@wits.ac.za (Y. Zelenyuk).

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topology on G with $\mathrm{Ult}(\mathcal{T}) = \{p\}$, then $\mathcal{T}$ is regular if and only if for every $q \in G^* \setminus \{p\}$, $qp \neq p$. Such idempotents are called *strongly right maximal*.

The most common finite subsemigroups of G^* are *bands*, that is, semigroups of idempotents. The simplest examples of bands are *left zero* semigroups, defined by the identity $xy = x$, *right zero* semigroups, defined by the identity $xy = y$, and *chains of idempotents*, with respect to the order $x \leqslant y$ if and only if $xy = yx = x$. The direct product of a left zero semigroup and a right zero semigroup is called a *rectangular* semigroup. Each band is a disjoint union of its maximal rectangular subsemigroups called *rectangular components* and these are partially ordered by the relation $P \leqslant Q$ if and only if $PQ \subseteq P$, equivalently $QP \subseteq P$.

An object P in some category is a *projective* if for every morphism $f : P \to Q$ and for every surjective morphism $g : R \to Q$, there exists a morphism $h : P \to R$ such that $g \circ h = f$. We say that an object P is an *absolute coretract* if for every surjective morphism $g : R \to P$ there exists a morphism $h : P \to R$ such that $g \circ h = \mathrm{id}_P$. Obviously, each projective is an absolute coretract. In many categories these notions coincide but not in all. Let $\mathfrak{F}$ and $\mathfrak{C}$ denote the categories of finite semigroups and compact Hausdorff right topological semigroups, respectively. Then the finite absolute coretracts in $\mathfrak{C}$ are the same as the projectives in $\mathfrak{F}$, and these are certain chains of rectangular bands, in particular, the finite left (right) zero semigroups and chains of idempotents are such [6].

For every regular almost maximal left invariant topology $\mathcal{T}$ on G, $S = \mathrm{Ult}(\mathcal{T})$ is a projective in $\mathcal{F}$ [7, Theorem 4.1]. Assuming Martin's Axiom (MA), for every finite absolute coretract S in $\mathfrak{C}$, there is a regular left invariant topology $\mathcal{T}$ on G with $\mathrm{Ult}(\mathcal{T})$ isomorphic to S, and in the case $G = \bigoplus_\omega \mathbb{Z}_2$, $\mathcal{T}$ can be chosen to be a group topology (that is, a topology making the group a topological group) [7, Theorem 5.2 and Lemma 6.10]. Every countable almost maximal topological group contains an open Boolean subgroup [9, Theorem 10.15], and its existence cannot be established in ZFC, the system of usual axioms of set theory [9, Corollary 10.17]. However, for every $n \in \mathbb{N}$, there is in ZFC a regular left invariant topology $\mathcal{T}$ on G with $\mathrm{Ult}(\mathcal{T})$ being a chain of n idempotents [7, Theorem 6.1].

Let $\mathcal{T}$ be a left invariant topology on G. A filter $\mathcal{F}$ on G is *$\mathcal{T}$-fundamental* if for every neighborhood U of 1, there is $x \in G$ such that $xU \in \mathcal{F}$. We say that $\mathcal{T}$ is *complete* if every $\mathcal{T}$-fundamental filter (equivalently, $\mathcal{T}$-fundamental nonprincipal ultrafilter) converges. Notice that in the case where $\mathcal{T}$ is a group topology, these notions coincide with the usual ones (with respect to the left uniformity).

Lemma 1.1. *Let $\mathcal{T}$ be a left invariant topology on G and let $S = \mathrm{Ult}(\mathcal{T})$. Then $\mathcal{T}$ is complete if and only if for every $p, q \in G^*$, $pq \in S$ implies that $q = aq_0$ for some $q_0 \in S$ and $a \in G$.*

Proof. Necessity. Let $p, q \in G^*$ and suppose that $pq \in S$. Then for every neighborhood U of 1 in $\mathcal{T}$, there is $x_U \in G$ such that $x_U q \in U$, equivalently $x_U^{-1} U \in q$, so q is $\mathcal{T}$-fundamental. Since $\mathcal{T}$ is complete, q converges to some $a \in G$. Hence, $q = aq_0$ for some $q_0 \in S$.

Sufficiency. Let q be a $\mathcal{T}$-fundamental nonprincipal ultrafilter on G. For every neighborhood U of 1 in $\mathcal{T}$, pick $x_U \in G$ such that $x_U U \in q$, equivalently $U \in x_U^{-1} q$. Let p be an ultrafilter on G extending the family of subsets

$$\left\{x_V^{-1} \colon\ V \text{ is a neighborhood of 1 contained in } U\right\},$$

where U runs over neighborhoods of 1. Then pq converges to 1, so $pq \in S$ and $p \in G^*$. But then $q = aq_0$ for some $q_0 \in S$ and $a \in G$. Hence, q converges to a. $\square$

Corollary 1.2. *Let $\mathcal{T}$ be an almost maximal left invariant topology on G and let $S = \mathrm{Ult}(\mathcal{T})$. Then $\mathcal{T}$ is regular and complete if and only if*

(1) *for every $a \in G \setminus \{1\}$, $(aS) \cap S = \emptyset$, and*
(2) *for every $p, q \in G^*$, $pq \in S$ implies that $q = bq_0$ and $p = p_0 b^{-1}$ for some $p_0, q_0 \in S$ and $b \in G$.*

Proof. Necessity. (1) means that $\mathcal{T}$ is Hausdorff [7, Proposition 2.5]. We need to check (2). Since $\mathcal{T}$ is complete, by Lemma 1.1, $q = bq_0$ for some $q_0 \in S$ and $b \in G$. Then since $\mathcal{T}$ is regular and $pbq_0 \in (pbS) \cap S$, $p_0 = pb \in S$, so $p = p_0 b^{-1}$.

Sufficiency. That $\mathcal{T}$ is complete follows from (2) and Lemma 1.1. To see that $\mathcal{T}$ is regular, suppose that $pq \in S$ for some $p \in \beta G$ and $q \in S$. If $p \in G$, then $p = 1$ by (1). Let $p \in G^*$. Then by (2), $q = bq_0$ and $p = p_0 b^{-1}$ for some $p_0, q_0 \in S$ and $b \in G$. Since $q \in S$, $b = 1$, so $p = p_0 \in S$. $\square$

For maximal left invariant topologies Corollary 1.2 simplifies, since in this case condition (1) is always satisfied (by [1, Theorem 3.34]).

We say that an idempotent $p \in G^*$ is *absolutely maximal* if for every $q, r \in G^*$, $p = qr$ implies that $r = ap$ and $q = pa^{-1}$ for some $a \in G$.

Corollary 1.3. *Let $\mathcal{T}$ be a maximal left invariant topology on G and let $\mathrm{Ult}(\mathcal{T}) = \{p\}$. Then $\mathcal{T}$ is regular and complete if and only if p is absolutely maximal.*