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Bounds for fixed points on Seifert manifolds

Qiang Zhang

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China

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ABSTRACT

In this paper, we consider homeomorphisms of compact connected orientable Seifert manifolds with hyperbolic orbifolds, and give some bounds involving the rank and the index of fixed point classes. One consequence is an index bound for fixed point classes. We rely on the classification of 2-orbifolds homeomorphisms and the similar bounds on surfaces.

1. Introduction

Fixed point theory studies fixed points of a selfmap f of a space X. Nielsen fixed point theory, in particular, is concerned with the properties of the fixed point set $\mathrm{Fix}(f)$ that are invariant under homotopy of the map f (see [4] for an introduction).

The fixed point set $\mathrm{Fix}(f)$ splits into a disjoint union of *fixed point classes*: two fixed points are in the same class if and only if they can be joined by a *Nielsen path*, which is a path homotopic (relative to endpoints) to its own f-image. For each fixed point class $\mathbf{F}$, a homotopy invariant *index* $\mathrm{ind}(f,\mathbf{F}) \in \mathbb{Z}$ is defined. A fixed point class is *essential* if its index is non-zero.

In a paper by B.J. Jiang, S.D. Wang and Q. Zhang [9], another homotopy invariant *characteristic* $\mathrm{chr}(f,\mathbf{F}) \in \mathbb{Z}$ is defined for a fixed point class $\mathbf{F}$. For an endomorphism $\phi: G \to G$ of a group G, its *fixed subgroup* refers to the subgroup $\mathrm{Fix}(\phi) := \{g \in G \mid g = \phi(g)\} \subset G$. The *stabilizer* of a fixed point $x \in \mathbf{F}$ is the subgroup $\mathrm{Stab}(f,x) := \mathrm{Fix}(f_\pi) \subset \pi_1(X,x)$, where $f_\pi: \pi_1(X,x) \to \pi_1(X,x)$ is the induced endomorphism. Since it is independent of the choice of $x \in \mathbf{F}$, up to isomorphism, the *stabilizer* of a fixed point class $\mathbf{F}$ is defined as $\mathrm{Stab}(f,\mathbf{F}) := \mathrm{Stab}(f,x)$, for any $x \in \mathbf{F}$. The *rank* of $\mathbf{F}$ is defined as

$$\mathrm{rank}(f,\mathbf{F}) := \mathrm{rank}\,\mathrm{Stab}(f,\mathbf{F}),$$

where the rank of a group is the minimal number of generators. The *characteristic* of a fixed point class $\mathbf{F}$ is defined as

$$\mathrm{chr}(f,\mathbf{F}) := 1 - \mathrm{rank}(f,\mathbf{F}),$$

E-mail address: zhangq.math@mail.xjtu.edu.cn.

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with the exception that when $\mathrm{Stab}(f, \mathbf{F}) = \pi_1(S)$ for some closed hyperbolic surface $S \subset X$,

$$\mathrm{chr}(f, \mathbf{F}) := \chi(S) = 2 - \mathrm{rank}(f, \mathbf{F}).$$

For brevity, we will write $\mathrm{Stab}(\mathbf{F})$, $\mathrm{rank}(\mathbf{F})$ and $\mathrm{chr}(\mathbf{F})$ if no confusion exists for the selfmap f in the context.

What is more, they gave the following bounds involving the characteristic and the index of fixed point classes.

Theorem 1.1. *([9, Theorem 1.1]) Suppose X is either a connected finite graph or a connected compact hyperbolic surface, and $f : X \to X$ is a selfmap. Then*

(A) $\mathrm{ind}(\mathbf{F}) \leqslant \mathrm{chr}(\mathbf{F})$ *for every fixed point class* $\mathbf{F}$ *of* f;
(B) *when X is not a tree,*

$$\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0} \bigl\{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\bigr\} \geqslant 2\chi(X),$$

where the sum is taken over all fixed point classes $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$.

In this paper, we consider homeomorphisms of compact connected orientable Seifert manifolds, and give similar bounds involving the rank and the index of fixed point classes. Our main result is

Theorem 1.2. *Suppose M is a compact connected orientable Seifert manifold (closed or with boundary) with hyperbolic orbifold $X(M)$, and $f : M \to M$ is a homeomorphism. Then*

(A) $\mathrm{ind}(\mathbf{F}) \leqslant \mathrm{chr}(\mathbf{F})$ *for every essential fixed point class* $\mathbf{F}$ *of* f;
(B) $\sum_{\mathrm{ind}(\mathbf{F})+\mathrm{chr}(\mathbf{F})<0}\{\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F})\} \geqslant \mathcal{B}$, *where the sum is taken over all essential fixed point classes* $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + \mathrm{chr}(\mathbf{F}) < 0$, *and*

$$\mathcal{B} = \begin{cases} 4(3 - \mathrm{rank}\,\pi_1(M)) & M = F \times S^1, \text{ where } F \text{ is a closed surface}, \\ 4(2 - \mathrm{rank}\,\pi_1(M)) & \text{others}. \end{cases}$$

As a direct consequence, we have bounds on the indices of fixed point classes.

Theorem 1.3. *Suppose M is a compact connected orientable Seifert manifold with hyperbolic orbifold $X(M)$, and $f : M \to M$ is a homeomorphism. Then the index of each fixed point class of f has bounds. More precisely,*

(A) $\mathrm{ind}(\mathbf{F}) \leqslant 1$ *for every fixed point class* $\mathbf{F}$ *of* f;
(B) *Let $\mathcal{B}$ be defined as in Theorem 1.2. Then*

$$\sum_{\mathrm{ind}(\mathbf{F})+1<0} \bigl\{\mathrm{ind}(\mathbf{F}) + 1\bigr\} \geqslant \mathcal{B},$$

where the sum is taken over all fixed point classes $\mathbf{F}$ *with* $\mathrm{ind}(\mathbf{F}) + 1 < 0$.

The bounds are similar to the one on graphs and surfaces [5]. When f is orientation preserving, B.J. Jiang and S.C. Wang [7] proved that the index of each essential fixed point class of f is ± 1.

Moreover, we can get an immature bound on the rank of fixed subgroups from Theorem 1.2 immediately.

Corollary 1.4. *Suppose $f : M \to M$ is a homeomorphism of a compact connected orientable Seifert manifold with hyperbolic orbifold $X(M)$. Let $f_\pi : \pi_1(X, x) \to \pi_1(X, x)$ be the induced automorphism and $\mathrm{Fix}(f_\pi) := \{\gamma \in \pi_1(X, x) \mid \gamma = f_\pi(\gamma)\} \subset \pi_1(X, x)$, where x is in an essential fixed point class. Then*

$$\mathrm{rank}\,\mathrm{Fix}(f_\pi) < 2\,\mathrm{rank}\,\pi_1(M).$$

Remark. In a recent paper by J.F. Lin and S.C. Wang [10], it is proved that for each automorphism $\phi : \pi_1(M) \to \pi_1(M)$ on a hyperbolic 3-manifold M, $\mathrm{rank}\,\mathrm{Fix}(\phi) < 2\,\mathrm{rank}\,\pi_1(M)$. However, If $\phi : \pi_1(M) \to \pi_1(M)$ is a generic automorphism of a Seifert manifold M, then the similar result in Corollary 1.4 dose not hold, and S.C. Wang give a counter example (Example 5.2) such that the rank of the fixed subgroup is infinite.

Notations and conventions. Given a set X in M, we use $N(X)$ to denote a regular neighborhood of X, use $\#X$ to denote the number of points in X.

A map f on a Seifert fiber space M is a *fiber preserving map* if it maps fibers to fibers. Identifying each fiber of M to a point, we get a set $X(M)$, which has a natural 2-dimensional orbifold structure [12, §3]. If f is fiber preserving, it induces a map $f' : X(M) \to X(M)$. An isotopy h_t $(t \in I)$ of a Seifert fiber space is *fiber preserving* if each h_t is a fiber preserving map. It is a *fiberwise isotopy* if h_t maps each fiber to itself.