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Cleavability over scattered first-countable LOTS

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ABSTRACT

In this paper we will show that if X is a compactum cleavable over a first-countable scattered linearly ordered topological space (LOTS) Y, then X does not have to be homeomorphic to a subspace of Y. We will then discover the conditions under which cleavability implies a homeomorphism exists. Furthermore, we will show that if X is a compactum cleavable over a first-countable LOTS, then X is a LOTS.

1. Introduction

A space X is said to be *cleavable* over a space Y *along* $A \subseteq X$ if there exists a continuous $f : X \to Y$ such that $f(A) \cap f(X \setminus A) = \emptyset$. A space X is *cleavable over* Y if it is cleavable over Y along all $A \subseteq X$. The topic was introduced by A.V. Arhangel'skiĭ and D.B. Shakhmatov in [1], though it was originally termed *splitting*, and it was in [2] that A.V. Arhangel'skiĭ posed the main questions related to the study of cleavability:

Question 1. *When does cleavability of a space X over a Hausdorff space Y imply the existence of a homeomorphism from X to a subspace of Y?*

Question 2. *Let X be a compactum cleavable over a linearly ordered topological space (LOTS). Is X a LOTS?*

For an answer to the first question to be anything but trivial, one must impose some restrictions on X and Y. Firstly, X and Y must be infinite. The following is an example of a trivial result.

Example 1.1. Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, both with the discrete topologies. Then X is cleavable over Y, but not homeomorphic to a subspace of Y.

The same results are reached for any discrete X and finite Y such that $|X| > |Y| > 1$.

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In the initial research in this area, authors have often considered the special case where Y is a LOTS. (See [2–4].) The most significant consequences are reached, however, when one assumes X is compact. Every continuous function from a compact space X to Hausdorff space Y must be closed, and therefore, proving a homeomorphism from X to a subspace of Y exists becomes proving the existence of a continuous injective function from X into Y.

Including this information into our exploration of cleavability, let us restate the question on which we will be focusing:

Question 3. *For infinite spaces X and Y, does cleavability of a compact space X over a LOTS Y imply the existence of a homeomorphism from X to a subspace of Y?*

This paper examines cleavability over scattered first-countable LOTS, specifically cleavability over ω_1. In Theorem 3.1, however, we provide a negative answer to Question 3; that is, we give a simple example of an infinite compact space X and an infinite LOTS Y such that X is cleavable over Y, but not embeddable into Y. We then prescribe the exact conditions under which cleavability of a compactum X over a first-countable LOTS Y implies such a homeomorphism exists. For papers containing other results on cleavability, see [2] and [4].

Additionally, there are many papers dedicated to showing when a space is linearly orderable, and what properties linearly orderable spaces have (see [5] and [6] for examples). The results of this paper provide a new characterization for when compacta are linearly orderable: that is, an infinite compactum is linearly orderable if it is cleavable over a first-countable scattered LOTS (Theorem 2.15).

In Section 2 we derive basic properties a compactum X must have when it is cleavable over a scattered first-countable LOTS Y. In particular, we prove that any compactum cleavable over ω_1 must be embeddable into ω_1 (Theorem 2.14). The most important and interesting results of this paper, however, are contained in Section 3: we begin by providing a counter-example, in which cleavability does not imply embeddability (Theorem 3.1). Then we describe the necessary conditions under which cleavability of a compactum X over a first-countable scattered linearly ordered topological space (LOTS) Y implies embeddability of X into Y (Theorem 3.16).

2. General theorems

We commence our exploration by discussing those properties an infinite compactum X must have if it is cleavable over a scattered first-countable LOTS. We then use these properties to prove the main theorems of this section, Theorems 2.13 and 2.14: that any infinite compactum cleavable over ω_1 must embed into ω_1, and must also therefore be a LOTS.

To begin, we state two introductory theorems, the first of which is from [2].

Theorem 2.1. *If X is cleavable over a Hausdorff space Y, then X is Hausdorff.*

Theorem 2.2. *If X is a compactum cleavable over a scattered LOTS Y, then X is also scattered.*

Proof. Assume for a contradiction that X contains a dense in itself subset D, and consider $\overline{D}$. We know $\overline{D}$ is compact T_2, and perfect, therefore by [7] it is resolvable; that is, $\overline{D} = A \cup B$, $\overline{A} = \overline{B} = \overline{D}$, and $A \cap B = \emptyset$. Then no function can cleave apart A and B. Therefore X cannot contain a dense in itself subset. □

The following results and definition can be found in [8] (Lemma 2.3 and Theorem 2.4), and [9] (Definition 2.5 and Theorem 2.6).

Lemma 2.3. *If X is cleavable over Y, and Y is cleavable over Z, then X is cleavable over Z.*

Theorem 2.4. *If X is a compactum cleavable over a first-countable LOTS Y, then X is first-countable.*

Definition 2.5. A space X is said to be **Fréchet–Urysohn** if for every $A \subseteq X$ and $x \in \overline{A}$, there exists a sequence contained in A that converges to X.

Lemma 2.6. *Every first-countable space X is Fréchet–Urysohn.*

In general, the implication of Lemma 2.6 is not reversable. However, it is important to note that as we are dealing with first-countable X and Y, we will be relying on the property described in Definition 2.5 for many of the following proofs. One should also note that every LOTS Y is normal, therefore Hausdorff and regular. Further, if X is compact and T_2, then it is also regular and normal.

The following two theorems may be found in [10] and [9] respectively.

Theorem 2.7. *Every first-countable compact scattered space is metrizable.*