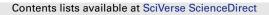
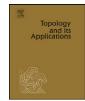
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# Cleavability over scattered first-countable LOTS

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#### 1. Introduction

## ABSTRACT

In this paper we will show that if X is a compactum cleavable over a first-countable scattered linearly ordered topological space (LOTS) Y, then X does not have to be homeomorphic to a subspace of Y. We will then discover the conditions under which cleavability implies a homeomorphism exists. Furthermore, we will show that if X is a compactum cleavable over a first-countable LOTS, then X is a LOTS.

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A space *X* is said to be *cleavable* over a space *Y* along  $A \subseteq X$  if there exists a continuous  $f : X \to Y$  such that  $f(A) \cap f(X \setminus A) = \emptyset$ . A space *X* is *cleavable* over *Y* if it is cleavable over *Y* along all  $A \subseteq X$ . The topic was introduced by A.V. Arhangel'skiĭ and D.B. Shakhmatov in [1], though it was originally termed *splitting*, and it was in [2] that A.V. Arhangel'skiĭ posed the main questions related to the study of cleavability:

**Question 1.** When does cleavability of a space X over a Hausdorff space Y imply the existence of a homeomorphism from X to a subspace of Y?

**Question 2.** Let X be a compactum cleavable over a linearly ordered topological space (LOTS). Is X a LOTS?

For an answer to the first question to be anything but trivial, one must impose some restrictions on X and Y. Firstly, X and Y must be infinite. The following is an example of a trivial result.

**Example 1.1.** Let  $X = \{x_1, x_2, x_3\}$  and  $Y = \{y_1, y_2\}$ , both with the discrete topologies. Then X is cleavable over Y, but not homeomorphic to a subspace of Y.

The same results are reached for any discrete *X* and finite *Y* such that |X| > |Y| > 1.

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Including this information into our exploration of cleavability, let us restate the question on which we will be focusing:

**Question 3.** For infinite spaces X and Y, does cleavability of a compact space X over a LOTS Y imply the existence of a homeomorphism from X to a subspace of Y?

This paper examines cleavability over scattered first-countable LOTS, specifically cleavability over  $\omega_1$ . In Theorem 3.1, however, we provide a negative answer to Question 3; that is, we give a simple example of an infinite compact space *X* and an infinite LOTS *Y* such that *X* is cleavable over *Y*, but not embeddable into *Y*. We then prescribe the exact conditions under which cleavability of a compactum *X* over a first-countable LOTS *Y* implies such a homeomorphism exists. For papers containing other results on cleavability, see [2] and [4].

Additionally, there are many papers dedicated to showing when a space is linearly orderable, and what properties linearly orderable spaces have (see [5] and [6] for examples). The results of this paper provide a new characterization for when compacta are linearly orderable: that is, an infinite compactum is linearly orderable if it is cleavable over a first-countable scattered LOTS (Theorem 2.15).

In Section 2 we derive basic properties a compactum *X* must have when it is cleavable over a scattered first-countable LOTS *Y*. In particular, we prove that any compactum cleavable over  $\omega_1$  must be embeddable into  $\omega_1$  (Theorem 2.14). The most important and interesting results of this paper, however, are contained in Section 3: we begin by providing a counter-example, in which cleavability does not imply embeddability (Theorem 3.1). Then we describe the necessary conditions under which cleavability of a compactum *X* over a first-countable scattered linearly ordered topological space (LOTS) *Y* implies embeddability of *X* into *Y* (Theorem 3.16).

#### 2. General theorems

We commence our exploration by discussing those properties an infinite compactum *X* must have if it is cleavable over a scattered first-countable LOTS. We then use these properties to prove the main theorems of this section, Theorems 2.13 and 2.14: that any infinite compactum cleavable over  $\omega_1$  must embed into  $\omega_1$ , and must also therefore be a LOTS.

To begin, we state two introductory theorems, the first of which is from [2].

**Theorem 2.1.** If X is cleavable over a Hausdorff space Y, then X is Hausdorff.

**Theorem 2.2.** If X is a compactum cleavable over a scattered LOTS Y, then X is also scattered.

**Proof.** Assume for a contradiction that *X* contains a dense in itself subset *D*, and consider  $\overline{D}$ . We know  $\overline{D}$  is compact  $T_2$ , and perfect, therefore by [7] it is resolvable; that is,  $\overline{D} = A \cup B$ ,  $\overline{A} = \overline{B} = \overline{D}$ , and  $A \cap B = \emptyset$ . Then no function can cleave apart *A* and *B*. Therefore *X* cannot contain a dense in itself subset.  $\Box$ 

The following results and definition can be found in [8] (Lemma 2.3 and Theorem 2.4), and [9] (Definition 2.5 and Theorem 2.6).

Lemma 2.3. If X is cleavable over Y, and Y is cleavable over Z, then X is cleavable over Z.

**Theorem 2.4.** If X is a compactum cleavable over a first-countable LOTS Y, then X is first-countable.

**Definition 2.5.** A space *X* is said to be **Fréchet–Urysohn** if for every  $A \subseteq X$  and  $x \in \overline{A}$ , there exists a sequence contained in *A* that converges to *X*.

Lemma 2.6. Every first-countable space X is Fréchet-Urysohn.

In general, the implication of Lemma 2.6 is not reversable. However, it is important to note that as we are dealing with first-countable X and Y, we will be relying on the property described in Definition 2.5 for many of the following proofs. One should also note that every LOTS Y is normal, therefore Hausdorff and regular. Further, if X is compact and  $T_2$ , then it is also regular and normal.

The following two theorems may be found in [10] and [9] respectively.

**Theorem 2.7.** Every first-countable compact scattered space is metrizable.

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