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Exactly n-resolvable topological expansions

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ABSTRACT

For a cardinal $\kappa > 1$, a space $X = (X, \mathcal{T})$ is *κ-resolvable* if X admits κ-many pairwise disjoint $\mathcal{T}$-dense subsets; $(X, \mathcal{T})$ is *exactly κ-resolvable* if it is κ-resolvable but not κ^+-resolvable.

The present paper complements and supplements the authors' earlier work, which showed for suitably restricted spaces $(X, \mathcal{T})$ and cardinals $\kappa \geqslant \lambda \geqslant \omega$ that $(X, \mathcal{T})$, if κ-resolvable, admits an expansion $\mathcal{U} \supseteq \mathcal{T}$, with $(X, \mathcal{U})$ Tychonoff if $(X, \mathcal{T})$ is Tychonoff, such that $(X, \mathcal{U})$ is μ-resolvable for all $\mu < \lambda$ but is not λ-resolvable (cf. Comfort and Hu, 2010 [11, Theorem 3.3]). Here the "finite case" is addressed. The authors show in ZFC for $1 < n < \omega$: (a) every n-resolvable space $(X, \mathcal{T})$ admits an exactly n-resolvable expansion $\mathcal{U} \supseteq \mathcal{T}$; (b) in some cases, even with $(X, \mathcal{T})$ Tychonoff, no choice of $\mathcal{U}$ is available such that $(X, \mathcal{U})$ is regular (nor even quasi-regular); (c) if regular and n-resolvable, $(X, \mathcal{T})$ admits an exactly n-resolvable regular expansion $\mathcal{U}$ if and only if either $(X, \mathcal{T})$ is itself exactly n-resolvable or $(X, \mathcal{T})$ has a subspace which is either n-resolvable and nowhere dense or is $(2n)$-resolvable. In particular, every ω-resolvable regular space admits an exactly n-resolvable regular expansion. Further, for many familiar topological properties $\mathbb{P}$ (e.g., Tychonoff; has a clopen basis), one may choose $\mathcal{U}$ so that $(X, \mathcal{U}) \in \mathbb{P}$ if $(X, \mathcal{T}) \in \mathbb{P}$.

1. Introduction

Let $\kappa > 1$ be a (possibly finite) cardinal. Generalizing a concept introduced by Hewitt [18], Ceder [2] defined a space $(X, \mathcal{T})$ to be *κ-resolvable* if there is a family of κ-many pairwise disjoint nonempty subsets of X, each $\mathcal{T}$-dense in X. Generalizations of this concept (for example: the dense sets are perhaps not pairwise disjoint, but have pairwise intersections which are "small" in some sense; or, the dense sets are required to be Borel, or to be otherwise restricted), were introduced and studied in subsequent decades, for example in [24,25,5,6,26].

We refer the reader to such works as [22,9,23,11] for extensive bibliographic references relating to the existence of spaces, typically Tychonoff spaces, which satisfy certain prescribed resolvability properties but not others. The flavor of our work [11] is quite different from that of other papers known to us. In those papers, broadly speaking, the objective is either (a) to find conditions on a space sufficient to ensure some kind of resolvability or (b) to construct by *ad hoc* means spaces which for certain infinite cardinals λ are λ-resolvable (sometimes in a modified sense) but which are not κ-resolvable for specified $\kappa > \lambda$. In [11], in contrast, a broader spectrum of results is enunciated. We showed there that the tailor-made specific spaces constructed by those *ad hoc* arguments arise as instances of a widely available phenomenon, in this sense: *every* Tychonoff space satisfying mild necessary conditions admits larger Tychonoff topologies as in (b) above.

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The constructions of [11] are based on the $\mathcal{KID}$ expansion technique introduced in [19] and developed further in [20,7–9]. Roughly speaking, the present work in the finite context parallels theorems (cf. [11] (especially Theorem 3.3)) about κ-resolvability when κ is infinite. Specifically we show for fixed $n < \omega$ that every n-resolvable space admits an exactly n-resolvable expansion. In some cases, even when the initial space is Tychonoff, the expansion cannot be chosen to be quasi-regular. (See Definition 1.9 for the definition of this concept.) Further, we characterize explicitly those n-resolvable regular spaces which do admit an exactly n-resolvable quasi-regular expansion (Theorem 3.9). It is a pleasing feature of our arguments that for many familiar topological properties $\mathbb{P}$, when the initial hypothesized space $(X, \mathcal{T})$ has $\mathbb{P}$ and does admit an exactly n-resolvable quasi-regular expansion $\mathcal{U}$, one may arrange also that $(X, \mathcal{U})$ has $\mathbb{P}$.

Ad hoc constructions of Tychonoff spaces which for fixed $n < \omega$ are exactly n-resolvable have been available for some time [12]; see also [3,14,13,17], and [16] for other examples, not all Tychonoff.

Remark 1.1. In a preliminary version of this paper circulated to colleagues in August, 2010, we purported to have proved the statements claimed in our abstract [10] and [11, 5.4(**)]. We are grateful to an anonymous reader for indicating a simple counterexample (see Theorem 3.7 below for a broad generalization of the suggested argument); that example helped us to recognize the unavoidable relevance of the quasi-regularity property which figures prominently in this work, and to find the more delicate correct condition captured in Theorems 3.8 and 3.9 below.

We are grateful also to the referee of this paper for unusually thorough, careful and constructive comments. Among other contributions, s/he (a) identified a telling conceptual error on our part; (b) clarified, improved and even corrected our choice of wording at certain points; and (c) drastically shortened our proof of Lemma 3.6, by showing that preliminary consideration of a special case was unnecessary.

Following van Douwen [12], we call a space *crowded* if it has no isolated points. (Some authors prefer the term *dense-in-itself*.) Obviously every resolvable space is crowded.

Definition 1.2. Let $\kappa > 1$ be a (possibly finite) cardinal and let $X = (X, \mathcal{T})$ be a space. Then

(a) X is *hereditarily κ-irresolvable* if no nonempty subspace of X is κ-resolvable in the inherited topology;
(b) X is *hereditarily irresolvable* if X is hereditarily 2-irresolvable; and
(c) X is *open-hereditarily irresolvable* if no nonempty open subspace of X is resolvable in the inherited topology.

Notation 1.3. (a) Let $(X, \mathcal{T})$ be a space and let $Y \subseteq X$. The symbol $(Y, \mathcal{T})$ denotes the set Y with the topology inherited from $(X, \mathcal{T})$.

(b) Given a set X and $\mathcal{A} \subseteq \mathcal{P}(X)$, the smallest topology $\mathcal{T}$ on X such that $\mathcal{T} \supseteq \mathcal{A}$ is denoted $\mathcal{T} := \langle \mathcal{A} \rangle$.

It is proved in [4] that, in any space, the union of resolvable subsets is resolvable. The proof shows with minimal change that, for $\kappa > 1$, in any space the union of κ-resolvable subsets is κ-resolvable. (See also in this connection [13].) The following statement, included explicitly here at the suggestion of the referee, is then immediate.

Lemma 1.4. *Let $\kappa > 1$ be a (possibly finite) cardinal, let $X = (X, \mathcal{T})$ be a space, and set*

$$R = R(\kappa) := \bigcup \{S\colon (S, \mathcal{T}) \text{ is } \kappa\text{-resolvable}\}.$$

Then

(a) *If $R \neq \emptyset$ then R is κ-resolvable*;
(b) *R is closed in $(X, \mathcal{T})$*; *and*
(c) *$X \backslash R$ is hereditarily κ-irresolvable.*

Definition 1.5. Given $\kappa > 1$ and $X = (X, \mathcal{T})$ as in Lemma 1.4, the set $R = R(\kappa)$ is the *κ-resolvable hull* of X.

Remark 1.6. It is obvious that for $\kappa > 1$ and X a space, one has:

(a) $R = \emptyset$ if and only if X is hereditarily κ-irresolvable; and
(b) (from Lemma 1.4) $R = X$ if and only if X is κ-resolvable.

We say as usual that a topological property $\mathbb{P}$ is *closed-hereditary* [resp., *open-hereditary*; resp., *dense-hereditary*] if

$$\big[(X, \mathcal{T}) \in \mathbb{P},\ A \subseteq X,\ A \text{ closed [resp., open; resp., dense] in } X\big] \quad \Rightarrow \quad (A, \mathcal{T}) \in \mathbb{P}.$$

If, more generally, $[(X, \mathcal{T}) \in \mathbb{P},\ A \subseteq X] \Rightarrow (A, \mathcal{T}) \in \mathbb{P}$, then $\mathbb{P}$ is *hereditary*.