Topology and its Applications 159 (2012) 3426–3431

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Isometric embeddings of Polish ultrametric spaces ☆

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A R T I C L E I N F O

Article history:
Received 15 September 2011
Received in revised form 1 August 2012
Accepted 6 August 2012

MSC:
primary 54H15
secondary 20E34, 20E22

Keywords:
Polish metric spaces
Ultrametric spaces
Urysohn spaces
Isometry groups

A B S T R A C T

We characterize Polish ultrametric spaces all of whose isometric embeddings into the Polish ultrametric Urysohn space with the same set of distances are extensive, that is, they give rise to an embedding of their respective isometry groups. This generalizes a result proved by Gao and Shao (2011) [2].

1. Introduction

The Polish Urysohn space $\mathbb{U}$ is a unique Polish metric space that is universal in the class of Polish metric spaces, and ultrahomogeneous, that is, every isometric bijection between finite subsets of $\mathbb{U}$ can be extended to an isometry of $\mathbb{U}$. One of natural directions for exploration of the structure of the Urysohn space $\mathbb{U}$ is related to questions about types of subsets of $\mathbb{U}$ whose isometries can be extended to isometries of $\mathbb{U}$; and about how this can be done. For example, it is proved in [4] that every isometric bijection between compact subsets of $\mathbb{U}$ can always be extended to an isometry of $\mathbb{U}$.

In this note, we are interested in counterparts of $\mathbb{U}$ in the realm of Polish ultrametric spaces, that is, Polish metric spaces satisfying a strong version of the triangle inequality:

$$d(x,z) \leqslant \max\{d(x,y), d(y,z)\}.$$

It is easy to see that a separable ultrametric space may realize only countably many distances, so there is no chance for the existence of *the* Polish ultrametric Urysohn space. However, for any countable set $R \subseteq \mathbb{R}^{>0}$ there exists a Polish ultrametric space X_R that is ultrahomogeneous and universal for Polish ultrametric spaces all of whose nonzero distances are in R.

In [2], the authors introduced the notion of *extensive isometric embedding* of a metric space, which considerably strengthens simple extendability of isometries.

☆ This research was supported by the Marie Curie IRG grant REDUCE 224808.
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0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.08.006

Definition. Let X, Y be metric spaces, and let $\mathrm{Iso}(X)$, $\mathrm{Iso}(Y)$ be their full isometry groups endowed with the pointwise convergence topology. An isometric embedding

$$e : X \to Y$$

is called *extensive* if there is a topological group embedding

$$\Phi : \mathrm{Iso}(X) \to \mathrm{Iso}(Y)$$

such that for every $\phi \in \mathrm{Iso}(X)$ we have

$$\Phi(\phi) \restriction e[X] = e \circ \phi \circ e^{-1}.$$

They proved (Theorem 6.17 of [2]) that *every* isometric embedding of a compact ultrametric space X with nonzero distances contained in a countable $R \subseteq \mathbb{R}^{>0}$ into the Polish ultrametric Urysohn space X_R is extensive. We improve their result by giving a full characterization of Polish ultrametric spaces with nonzero distances in a fixed set R, all of whose isometric embeddings into X_R are extensive.

2. Ultrametric spaces

Let us start with a short review of basic facts and definitions concerning ultrametric spaces. A metric space X is called *Polish* if it is separable and complete. It is called *ultrametric*, or *non-archimedean*, if it satisfies a strong version of the triangle inequality:

$$d(x,z) \leqslant \max\bigl\{d(x,y), d(y,z)\bigr\},$$

for all $x, y, z \in X$.

Typical examples of ultrametric spaces are

(i) the family $\mathbb{N}^{\mathbb{N}}$ of all sequences of natural numbers with metric d defined by

$$d(x,y) = \max\bigl\{2^{-n}\colon x(n) \neq y(n)\bigr\} \quad \text{for } x \neq y;$$

(ii) a valued field K with valuation $|\cdot| : K \to \mathbb{R}$ and metric

$$d(x,y) = |x-y|;$$

in particular the field $\mathbb{Q}_p$ of p-adic numbers is an ultrametric space;

(iii) the space of all rays in an $\mathbb{R}$-tree T starting from a fixed point $t \in T$ can be canonically given an ultrametric structure.

If X is an ultrametric space, then the notions of open and closed balls are somewhat misleading because all balls

$$B^X_{<r}(x), \quad B^X_{\leqslant r}(x)$$

for $x \in X$, $r \in \mathbb{R}$ are topologically closed. Hence, following an already accepted terminology, the former will be called ‘open’, and the latter ‘closed’ balls.

An easy to prove but fundamental property of ultrametric spaces is that if b_1, b_2 are balls in an ultrametric space X, either ‘open’ or ‘closed’, then

$$b_1 \subseteq b_2 \quad \text{or} \quad b_2 \subseteq b_1 \quad \text{or} \quad b_1 \cap b_2 = \emptyset.$$

It implies that

$$d(y,x) = d(x,z) \quad \text{or} \quad d(x,y) = d(y,z) \quad \text{or} \quad d(x,z) = d(z,y) \tag{1}$$

for every $x, y, z \in X$. This observation will be repetitively used in this note.

An ultrametric space X is called *spherically complete* if

$$\bigcap_n b_n \neq \emptyset$$

for every decreasing sequence $\{b_n\}_{n\in\mathbb{N}}$ of ‘closed’ balls in X. Otherwise, it is *non-spherically complete*. An *r-polygon*, where $r \in \mathbb{R}$, is a set $P \subseteq X$ such that $d(x,y) = r$ for every $x, y \in P$.

For a metric space X, its *set of distances* R is defined by

$$R = \bigl\{r \in \mathbb{R}\colon \exists x, y \in X \ \bigl(x \neq y \text{ and } d(x,y) = r\bigr)\bigr\}.$$