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Hierarchy of graph matchbox manifolds

Olga Lukina

Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom

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ABSTRACT

We study a class of graph foliated spaces, or graph matchbox manifolds, initially constructed by Kenyon and Ghys. For graph foliated spaces we introduce a quantifier of dynamical complexity which we call its level. We develop the *fusion* construction, which allows us to associate to every two graph foliated spaces a third one which contains the former two in its closure. Although the underlying idea of the fusion is simple, it gives us a powerful tool to study graph foliated spaces. Using fusion, we prove that there is a hierarchy of graph foliated spaces at infinite levels. We also construct examples of graph foliated spaces with various dynamical and geometric properties.

1. Introduction

A *matchbox manifold* is a compact connected metrizable space M such that each point $x \in M$ has a neighborhood homeomorphic to a product space $U_x \times N_x$, where $U_x \subset \mathbb{R}^n$ is open and N_x is a compact totally disconnected space. The term 'matchbox manifold' originates from the works of Aarts and Martens [2], Aarts and Oversteegen [1] for the case when $n = 1$, when local charts can be thought of as 'boxes of matches'. The most well-studied classes of examples of matchbox manifolds are weak solenoids [25,15], generalized solenoids [35], and tiling spaces of aperiodic tilings with finite local complexity (see, for instance, [29], or [5] for a more general type of tilings). In this paper we consider a third class of examples, which we call *graph matchbox manifolds*. This construction was introduced by Kenyon and Ghys [16], and later generalized by Blanc [6], Lozano Rojo [22], Alcalde Cuesta, Lozano Rojo and Macho Stadler [3].

Remark 1.1. (On the use of terminology.) The notion of a matchbox manifold is essentially the same as that of a *lamination*. The term 'lamination' appears in the literature in two slightly different contexts: in low-dimensional topology, a lamination is a decomposition into leaves of a closed subset of a manifold; in holomorphic dynamics, Sullivan [33] introduced Riemann surface laminations as compact topological spaces locally homeomorphic to a complex disk times a Cantor set. An embedding into a manifold is not required in the latter context, and a matchbox manifold is a lamination in this terminology. The concept of a *foliated space* as a generalization of a foliated manifold was introduced in the book by Moore and Schochet [26], where a foliated space is defined as a separable metrizable space locally homeomorphic to a product of a disk in $\mathbb{R}^n$ and a separable metrizable space. In this terminology, a matchbox manifold is a foliated space with specific properties, i.e. it is a compact foliated space with totally disconnected transversals. In the present paper we follow terminology of Candel and

E-mail address: ollukina940@gmail.com.

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Conlon [8], reserving the word 'lamination' for a foliated space embedded in a manifold. We then use the term 'matchbox manifold' to distinguish a class of foliated spaces which are compact and have totally disconnected transversals.

Let G be a finitely generated group with a non-symmetric set of generators G_0, that is, if $h \in G_0$ then $h^{-1} \notin G_0$. Let $\mathcal{G}$ be the Cayley graph of G, and X be the set of all infinite connected subtrees of $\mathcal{G}$ containing the identity e. Each subtree T is equipped with a standard complete length metric d, and the pair (T, e) is a pointed metric space. The set X, endowed with the Gromov–Hausdorff metric d_{GH} [7], is a compact totally disconnected space [6,16,22]. One can define a partial action of the free group F_n on X, where n is the cardinality of the set of generators G_0. This action gives rise to a pseudogroup $\mathfrak{G}$ on X, and an important feature of the construction is that the pseudogroup dynamical system $(X, \mathfrak{G})$ can be realized as the holonomy pseudogroup of a smooth foliated space $\mathfrak{M}_G$ with two-dimensional leaves [6,16,22]. By this construction, for $(T, e) \in X$ the corresponding leaf $L_T \subset \mathfrak{M}_G$ can be thought of as the two-dimensional boundary of the thickening of a quotient graph of T, where the quotient map is determined by the geometry of T.

Definition 1.2. A *graph matchbox manifold* is the closure $\mathcal{M} = \bar{L}$ of a leaf L in $\mathfrak{M}_G$, that is, $\mathcal{M}$ is a closed saturated transitive subset of $\mathfrak{M}_G$.

1.1. Hierarchies of graph matchbox manifolds

In previous works the construction of Kenyon and Ghys was mostly used to produce examples of matchbox manifolds with specific geometric and ergodic properties. Ghys [16], see also [3], showed that if $G = \mathbb{Z}^2$ then $\mathfrak{M}_{\mathbb{Z}^2}$ contains a leaf L such that the matchbox manifold $\mathcal{M} = \bar{L}$ is minimal and has leaves with different conformal structures. Lozano Rojo [23] studied minimal examples in the case $G = \mathbb{Z}^2$ from the point of view of ergodic theory. In the case where $G = F_3$, a free group on three generators, Blanc [6] found an example of a graph matchbox manifold containing leaves with any possible number of ends.

In this paper we study a partial order on the corresponding foliated space $\mathfrak{M}_G$, given by inclusions. The following basic observation allows to restrict our attention to the case $G = F_n$ and the space of graph matchbox manifolds $\mathfrak{M}_n$.

Theorem 1.3. *Given a group G with a set of generators G_0 of cardinality at most n, there exists a foliated embedding*

$$\Phi : \mathfrak{M}_G \to \mathfrak{M}_n,$$

where $\mathfrak{M}_G$ and $\mathfrak{M}_n$ are foliated spaces obtained by the construction of Kenyon and Ghys for G and a free group F_n on n generators respectively.

Let $\mathcal{M}_1, \mathcal{M}_2 \subset \mathfrak{M}_n$ be graph matchbox manifolds, then the rule

$$\mathcal{M}_1 \preceq \mathcal{M}_2 \quad \text{if and only if} \quad \mathcal{M}_1 \subseteq \mathcal{M}_2$$

defines a partial order on the set $\mathcal{S}_n$ of graph matchbox manifolds in $\mathfrak{M}_n$. Compact leaves and minimal subsets of $\mathfrak{M}_n$ are minimal elements in $\mathcal{S}_n$ with respect to this order. The following theorem describes the structure of $\mathfrak{M}_n$.

Recall [6] that a leaf $L \subset \mathcal{M}$ is *recurrent* if and only if L is transitive and accumulates on itself. A leaf L is *proper* if it does not accumulate on itself.

Theorem 1.4. *The partially ordered set $(\mathcal{S}_n, \preceq)$ of graph matchbox manifolds in the foliated space $\mathfrak{M}_n$, $n > 1$, has the following properties.*

(1) *the set $C = \{L \subset \mathfrak{M}_n \mid L$ is compact$\}$ is a dense meager subset of $\mathfrak{M}_n$. Moreover, $C \cap \mathcal{X}$ is countable, where $\mathcal{X}$ is a canonical embedding of X into $\mathfrak{M}_n$.*
(2) *$(\mathcal{S}_n, \preceq)$ is a directed partially ordered set, i.e. given $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{S}_n$ there exists $\mathcal{M}_3 \in \mathcal{S}_n$ such that $\mathcal{M}_1 \cup \mathcal{M}_2 \subseteq \mathcal{M}_3$.*
(3) *$(\mathcal{S}_n, \preceq)$ contains a unique maximal element $\mathcal{M}_{max} = \mathfrak{M}_n$ which has a recurrent leaf. Therefore, $\mathfrak{M}_n$ contains a residual subset of recurrent leaves.*

In order to prove Theorem 1.4(2), we introduce the 'fusion' construction which associates to any two transitive subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathfrak{M}_n$ a transitive subset $\mathcal{M}_3$ such that $\mathcal{M}_3 \supseteq \mathcal{M}_1 \cup \mathcal{M}_2$. More precisely, given pointed graphs (T_1, e) and (T_2, e) such that $\mathcal{M}_1 = \overline{L_{T_1}}$ and $\mathcal{M}_2 = \overline{L_{T_2}}$ we give a recipe to construct a graph (T_3, e) such that $\mathcal{M}_3 = \overline{L_{T_3}}$ satisfies the required property. The underlying idea of the construction is very simple, but it gives us a powerful tool which allows us to obtain a lot of information about hierarchy and properties of graph matchbox manifolds. Theorems 1.4(2) and 1.4(3) are the first applications of fusion.

Theorem 1.4(2) is a direct consequence of the fusion. The next important observation is that in the space $\mathfrak{M}_n$ fusion enables us to construct infinite increasing chains of graph matchbox manifolds. Using [6, Theorem 3.5] with slightly eased assumptions, we conclude that the closure of such a chain contains a dense leaf, and if every element in the chain is distinct,