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# The semigroup of ultrafilters near an idempotent of a semitopological semigroup

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ABSTRACT

Let $(T,+)$ be a Hausdorff semitopological semigroup, $S$ be a dense subsemigroup of $T$ and $e$ be an idempotent element of $T$. The set $e^*_S$ of ultrafilters on $S$ that converge to $e$ is a semigroup under restriction of the usual operation $+$ on $\beta T_d$, the Stone–Čech compactification of the discrete semigroup $T_d$. We characterize the smallest ideal of $(e^*_S,+)$, and those sets "central" in $(e^*_S,+)$, that is, those sets which are members of minimal idempotents in $(e^*_S,+)$. We describe some combinatorial applications of those sets that are central in $(e^*_S,+)$.



## 1. Introduction

Given a discrete space $(T,+)$, we take the points of $\beta T$, the Stone–Čech compactification of $T$, to be the ultrafilters on $T$ (or the ultrafilters in the collection of all subsets of $T$), with the points of $T$ identified with the principal ultrafilters. The topology of $\beta T$ can be defined by stating that the sets of the form $\{p \in \beta T\colon\ A \in p\}$, where $A$ is a subset of $T$, are a base for the open sets. For any $p \in \beta T$ and any $A \subseteq T$, $A \in p$ if and only if $p \in \overline{A}$, where $\overline{A}$ denotes $cl_{\beta T} A$. If $A$ is a subset of $T$, we shall use $A^*$ to denote $\overline{A} - A$.

Let $p, q \in \beta T$ and $A \subseteq T$, then $A \in p+q$ if and only if $\{s \in T\colon\ -s+A \in q\} \in p$, where $-s+A = \{t \in T\colon\ s+t \in A\}$. For every $p \in \beta T$, the map $r_p : \beta T \to \beta T$ defined by $r_p(q) = q+p$ is continuous, for every $s \in T$, the map $\lambda_s : \beta T \to \beta T$ defined by $\lambda_s(q) = s+q$ is continuous. For more details see [11] and [2].

The ultrafilter semigroup $(0^+,+)$ of the topological semigroup $T = ((0,+\infty),+)$ consists of all nonprincipal ultrafilter on $T = (0,+\infty)$ converging to 0 and is a closed subsemigroup in the Stone–Čech compactification $\beta T_d$ of $T$ as a discrete semigroup. In [8], N. Hindman and I. Leader characterized the smallest ideal of $(0^+,+)$, its closure, and those sets "central" in $(0^+,+)$, that is, those sets which are members of minimal idempotents in $(0^+,+)$. They derived new combinatorial applications of those sets that are central in $(0^+,+)$. Related topics in [3] and [4] can be found.

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In Section 2 we define ultrafilters near a point and show that if $e \in T$ is an idempotent then

$$e^* = \left\{ p \in \beta T_d \colon e \in \bigcap_{A \in p} cl_T(A) \right\}$$

is a compact subsemigroup of $\beta T_d$. (Given a topological space $X$, the notation $X_d$ represents the set $X$ with the discrete topology.)

In Section 3 we characterize the members of the smallest ideal of $(e^*, +)$ and its closure. We also describe those subsets of $T$ that have idempotents in $(e^*, +)$ in their closure.

Especially important in the combinatorial applications have been the central sets, i.e. those sets with idempotents in the intersection of their closure with the smallest ideal. In Section 4 we describe sets that are central near $e$, and we derive combinatorial results from their existence.

## 2. Preliminary

In this paper, $(T, +)$ denotes a Hausdorff semitopological semigroup that is not necessarily commutative. $E(T)$ denotes the collection of all idempotents in $T$. For every $x \in T$, $\tau_x$ denotes the collection of all neighborhoods of $x$, where a set $U$ is called a neighborhood of $x \in T$ if $x \in int_T(U)$. For a set $A$, we write $\mathcal{P}_f(A)$ for the set of finite nonempty subsets of $A$ and $\mathcal{P}(A)$ denote the collection of all subsets of $A$.

**Definition 2.1.** Let $(T, +)$ be a semitopological semigroup and $S$ be a subsemigroup of $T$.

(a) Given $x \in T$, $x_S^* = \{p \in \beta S_d \colon x \in \bigcap_{A \in p} cl_T A\}$.
(b) $B(S) = \bigcup_{x \in T} x_S^*$.
(c) $\infty^* = \beta S_d \setminus B(S)$.

The set $B(S)$ is the set of "bounded" ultrafilters on $S$ and the set $\infty^*$ is the set of "unbounded" ultrafilters on $S$.

We say $p \in x^*$ is a near point to $x$. If $x$ be a limit point of $T$, then $x^* \cap T^* \neq \emptyset$.

**Lemma 2.2.** *Let $(T, +)$ be a Hausdorff semitopological semigroup and $S$ be a semigroup of $T$.*

(a) *Let $x \in T$. Then $x_S^* \neq \emptyset$ if and only if $x \in cl_T(S)$.*
(b) *$p \in \infty^*$ if and only if for each $x \in T$ there exists $A \in p$ such that $x \notin cl_T(A)$.*
(c) *If $\tau_x \subseteq p$ then $p \in x_S^*$.*
(d) *$U$ is a neighborhood of $x$ if and only if $U \in p$ for each $p \in x_S^*$.*
(e) *Let $A \subseteq T$. Then $x \in cl_T A$ if and only if $cl_{\beta T_d} A \cap x_S^* \neq \emptyset$.*

**Proof.** (a) It is obvious.

(b) Let $p \in \infty^*$, so $p \notin x^*$ for each $x \in T$. Hence $x \notin \bigcap_{A \in p} cl_T A$. Thus $x \notin cl_T A$ for some $A \in p$.

Conversely, suppose for each $x \in T$, there exists $A \in p$ such that $x \notin cl_T A$. Hence $p \notin B(T)$ and thus $p \in \infty^*$.

(c) Let $U \in p$ for each $U \in \tau_x$, thus $U \cap A \neq \emptyset$ for each $U \in \tau_x$ and for each $A \in p$. This implies $x \in cl_T A$ for each $A \in p$. Therefore $p \in x^*$.

(d) Let $U \in \tau_x$ and $p \in x^*$. $U \cap A \neq \emptyset$ for each $A \in p$, so $U \in p$.

Conversely, let $x \in \partial U = U - int_T(U)$, so $\{U^c \cup \partial U\} \cup \tau_x \subseteq p$, thus $p \in x^*$ and $U \notin p$, a contradiction.

(e) It is obvious. □

**Lemma 2.3.** *Let $(T, +)$ be a semitopological semigroup and $S$ be a dense subsemigroup of $T$, then*:

(i) *For each $x, y \in T$, $x_S^* + y_S^* \subseteq (x + y)_S^*$.*
(ii) *Let $e \in T$ be an idempotent, then $e_S^*$ is a compact subsemigroup of $\beta S_d$.*

**Proof.** (i) Pick $p \in x_S^*$ and pick $q \in y_S^*$. Suppose that $A \in p + q$, then $\{t \in S \colon -t + A \in q\} \in p$. Since $-t + A \in q$ so $y \in cl_T(-t + A)$. Therefore there exists a net $\{x_\alpha\} \subseteq -t + A$ such that $x_\alpha \to y$ and so $t + x_\alpha \to t + y$ for each $t \in \{s \in S \colon -s + A \in q\}$. Also since $x \in cl_T(\{t \in S \colon -t + A \in q\})$, so there exists a net $\{t_\beta\}$ such that $t_\beta \to x$. Thus $\lim_\beta \lim_\alpha t_\beta + x_\alpha = x + y$. Since for each $\alpha$ and $\beta$, $t_\beta + x_\alpha \in A$ so $x + y \in cl_T(A)$. This completes the proof.

(ii) It is obvious. □

## 3. Idempotents and the smallest ideal of $e^*$

Let $S$ be a dense subsemigroup of a semitopological semigroup $(T, +)$, and $e \in E(T)$. Among the consequences of the fact that $(e_S^*, +)$ is a compact right topological semigroup is the fact that it contains idempotents [5, Corollary 2.10]. If $S$