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Leader type contractions, periodic and fixed points and new completivity in quasi-gauge spaces with generalized quasi-pseudodistances

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ABSTRACT

Leader's fixed point theorem – being more general as some Banach, Boyd and Wong, Browder, Burton, Caccioppoli, Dugundji and Granas, Geraghty, Krasnosel'skiĭ et al., Matkowski, Meir and Keeler, Mukherjea, Rakotch, Tasković, Walter and others' results – have played a great role in metric fixed point theory; in the literature the investigations of periodic points of contractions of Leader or Leader type are not known. We want to show how the introduced here *generalized quasi-pseudodistances* in quasi-gauge spaces can be used, in a natural way, to define contractions of Leader type and to obtain, for these contractions, the periodic and fixed point theorems without Hausdorff and sequentially complete assumptions about these spaces and without complete graph assumptions about these contractions, which was not done in the previous publications on this subject. The definitions, results and methods presented here are new for maps in quasi-gauge, topological, quasi-pseudometric and quasi-metric spaces. Examples are provided.

1. Introduction

The necessity of defining the various concepts of completivity in quasi-gauge spaces became apparent with the investigation of asymmetric structures in these spaces. General results of this sort were progressively shown in a series of papers and important ideas are to be found in [1,7,15,26,25,27,28], which also contains many examples. Also note that the studies of asymmetric structures and their applications in theoretical computer science are important.

We recall the definition of quasi-gauge spaces.

Definition 1.1. Let X be a nonempty set.

(i) A *quasi-pseudometric* on X is a map $p : X \times X \to [0, \infty)$ such that:

$(\mathcal{P}1)$ $\forall_{x \in X}\{p(x, x) = 0\}$; and
$(\mathcal{P}2)$ $\forall_{x,y,z \in X}\{p(x, z) \leqslant p(x, y) + p(y, z)\}$.

For given quasi-pseudometric p on X a pair (X, p) is called *quasi-pseudometric space*. A quasi-pseudometric space (X, p) is called *Hausdorff* if $\forall_{x,y \in X}\{x \neq y \Rightarrow p(x, y) > 0 \vee p(y, x) > 0\}$.

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(ii) Each family $\mathcal{P} = \{p_\alpha\colon \alpha \in \mathcal{A}\}$ of quasi-pseudometrics $p_\alpha : X \times X \to [0, \infty)$, $\alpha \in \mathcal{A}$, is called a *quasi-gauge* on X.

(iii) Let the family $\mathcal{P} = \{p_\alpha\colon \alpha \in \mathcal{A}\}$ be a quasi-gauge on X. The topology $\mathcal{T}(\mathcal{P})$, having as a subbase the family $\mathcal{B}(\mathcal{P}) = \{B(x, \varepsilon_\alpha)\colon x \in X, \ \varepsilon_\alpha > 0, \ \alpha \in \mathcal{A}\}$ of all balls $B(x, \varepsilon_\alpha) = \{y \in X\colon p_\alpha(x, y) < \varepsilon_\alpha\}$, $x \in X$, $\varepsilon_\alpha > 0$, $\alpha \in \mathcal{A}$, is called the topology *induced* by $\mathcal{P}$ on X.

(iv) (Dugundji [7], Reilly [25,26]) A topological space $(X, \mathcal{T})$ such that there is a quasi-gauge $\mathcal{P}$ on X with $\mathcal{T} = \mathcal{T}(\mathcal{P})$ is called a *quasi-gauge space* and is denoted by $(X, \mathcal{P})$.

(v) A quasi-gauge space $(X, \mathcal{P})$ is called *Hausdorff* if quasi-gauge $\mathcal{P}$ has the property: $\forall_{x,y\in X}\{x \neq y \Rightarrow \exists_{\alpha\in\mathcal{A}}\{p_\alpha(x, y) > 0 \vee p_\alpha(y, x) > 0\}\}$.

Remark 1.1. Each quasi-uniform space and each topological space is a quasi-gauge space (Reilly [25, Theorems 4.2 and 2.6]). The quasi-gauge spaces are the greatest general spaces with asymmetric structures.

By *Fix*(T) we denote the set of all *fixed points* of $T : X \to X$, i.e., *Fix*$(T) = \{w \in X\colon w = T(w)\}$. By a *contractive fixed point* of $T : X \to X$ we mean a point $w \in$ *Fix*(T) such that, for each $w^0 \in X$, $\lim_{m\to\infty} T^{[m]}(w^0) = w$.

In 1983 Leader [17, Theorem 3] discovered the following interesting phenomenon in the metric fixed point theory.

Theorem 1.1. *Let (X, d) be a metric space and let $T : X \to X$ be a map with a complete graph (i.e. closed in Y^2 where Y is the completion of X). The following hold:*

(a) *T has a contractive fixed point if and only if* (L1) $\forall_{x,y\in X}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{i,j\in\mathbb{N}}\{d(T^{[i]}(x), T^{[j]}(y)) < \varepsilon + \eta \Rightarrow d(T^{[i+r]}(x), T^{[j+r]}(y)) < \varepsilon\}$.
(b) *T has a fixed point if and only if* (L2) $\exists_{x\in X}\forall_{\varepsilon>0}\exists_{\eta>0}\exists_{r\in\mathbb{N}}\forall_{i,j\in\mathbb{N}}\{d(T^{[i]}(x), T^{[j]}(x)) < \varepsilon + \eta \Rightarrow d(T^{[i+r]}(x), T^{[j+r]}(x)) < \varepsilon\}$. *Moreover, if x, ε, η and r are as in* (L2) *and if* $\lim_{m\to\infty} T^{[m]}(x) = w$, *then* $\forall_{i\in\mathbb{N}}\{d(T^{[i]}(x), T^{[i+r]}(x)) \leqslant \eta \Rightarrow d(T^{[i+r]}(x), w) \leqslant \varepsilon\}$.

Recall, that the maps satisfying the above conditions (L1) and (L2) are called in literature *Leader contractions* and *weak Leader contractions*, respectively.

It is well-known that in complete metric spaces this result generalizes Banach [2], Boyd and Wong [3], Browder [4], Burton [5], Caccioppoli [6], Dugundji [7], Dugundji and Granas [8], Geraghty [9,10], Krasnosel'skiĭ et al. [16], Matkowski [19–21], Meir and Keeler [22], Mukherjea [23], Rakotch [24], Tasković [32], Walter [35] and many others' results not mentioned in this paper; for details, see, Jachymski [11,12] and Jachymski and Jóźwik [13]. In the complete metric spaces with τ-distances, beautiful generalizations of Leader's result [17, Theorem 3] are established by Suzuki [29, Theorem 4] and [30].

Recently, we introduced the concept of *generalized pseudodistances* in uniform spaces and showed that they provide a natural tool to obtain a natural generalizations of the results of [17, Theorem 3], [29, Theorem 4] and [30] in uniform spaces without sequentially complete assumptions and without complete graph assumptions about maps; for details see [36] and examples therein. These generalized pseudodistances generalize metrics, w-distances of Kada et al. [14], τ-functions of Lin and Du [18], τ-distances of Suzuki [31] and distances of Tataru [33] in metric spaces and distances of Vályi [34] in uniform spaces; for details, see [37,38].

In this paper, we show how the introduced here *generalized quasi-pseudodistances* in quasi-gauge spaces can be used, in a natural way, to define contractions of Leader type and to obtain for these contractions periodic and fixed point theorems without Hausdorff and sequentially complete assumptions about these spaces and without complete graph assumptions about these contractions.

It remains to note that, in the literature, the investigations of periodic points of contractions of Leader or Leader type are not known.

This paper is a continuation of [36–41].

2. Definitions and notations

We first record the definition of left (right) $\mathcal{J}$-families needed in the sequel.

Definition 2.1. Let $(X, \mathcal{P})$ be a quasi-gauge space. The family $\mathcal{J} = \{J_\alpha\colon \alpha \in \mathcal{A}\}$ of maps $J_\alpha : X \times X \to [0, \infty)$, $\alpha \in \mathcal{A}$, is said to be a *left* (*right*) *$\mathcal{J}$-family of generalized quasi-pseudodistances on X* (*left* (*right*) *$\mathcal{J}$-family on X*, for short) if the following two conditions hold:

($\mathcal{J}1$) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y,z\in X}\{J_\alpha(x, z) \leqslant J_\alpha(x, y) + J_\alpha(y, z)\}$; and
($\mathcal{J}2$) For any sequences $(u_m\colon m \in \mathbb{N})$ and $(v_m\colon m \in \mathbb{N})$ in X satisfying

$$\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_m, u_n) = 0\Big\} \tag{2.1}$$

$$\Big(\forall_{\alpha\in\mathcal{A}}\Big\{\lim_{m\to\infty}\sup_{n>m} J_\alpha(u_n, u_m) = 0\Big\}\Big) \tag{2.2}$$