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Basic subtoposes of the effective topos

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ABSTRACT

We study the lattice of local operators in Hyland's *Effective Topos*. We show that this lattice is a free completion under internal sups indexed by the natural numbers object, generated by what we call *basic* local operators.
We produce many new local operators and we employ a new concept, *sight*, in order to analyze these.
We show that a local operator identified by A.M. Pitts in his thesis, gives a subtopos with classical arithmetic.

0. Introduction

A fundamental concept in Topos Theory is the notion of *subtopos*: a subtopos of a topos $\mathcal{E}$ is a full subcategory which is closed under finite limits in $\mathcal{E}$, and such that the inclusion functor has a left adjoint which preserves finite limits. It then follows that this subcategory is itself a topos, and its internal logic has a convenient description in terms of the internal logic of $\mathcal{E}$. Subtoposes of $\mathcal{E}$ are in 1–1 correspondence with *local operators* in $\mathcal{E}$: these are certain endomaps on the subobject classifier of $\mathcal{E}$.

Whereas local operators/subtoposes of Grothendieck toposes can be neatly described in terms of Grothendieck topologies, for realizability toposes the study of local operators is not so easy. Yet it is important, since many variations on realizability, such as modified realizability, extensional realizability and Lifschitz realizability arise as the internal logic of subtoposes of standard realizability toposes.

Already in his seminal paper [2] where he introduces the effective topos $\mathcal{E}\!f\!f$ (the mother of all realizability toposes), Martin Hyland studied local operators and established that there is an order-preserving embedding of the Turing degrees in the lattice of local operators. Part of the groundwork for this treatment was laid by Andy Pitts in his thesis [13]. Moreover, Pitts exhibits a local operator which is different from Hyland's examples; this local operator will be studied also in the present paper. Wesley Phoa [12] has an alternative description of Hyland's "relative computability" local operators. Matías Menni [9] develops some general results on local operators in exact completions (such as $\mathcal{E}\!f\!f$). Finally, the second author of the present paper identified the local operator which corresponds to Lifschitz' realizability [19,20]. But as far as we are aware, this is all.

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The lattice of local operators in $\mathcal{E}\!f\!f$ is vast and notoriously difficult to study. We seem to lack methods to construct local operators and tell them apart. The present paper aims to improve on this situation in the following way: it is shown (Theorem 2.4) that every local operator is the internal join of a family (indexed by a nonempty set of natural numbers) of local operators induced by a nonempty family of subsets of $\mathbb{N}$ (which we call *basic* local operators). Then, we introduce a technical tool (*sights*) by which we can study inequalities between basic local operators. We construct an infinity of new basic local operators and we have some results about what new functions from natural numbers to natural numbers arise in the corresponding subtoposes. For many of our finitary examples (finite collections of finite sets) we can show that they do not create any new number-theoretic functions; for Pitts' example we can show that it forces all *arithmetical* functions to be total. This seems interesting: we have a realizability-like topos which, though far from being Boolean, nonetheless satisfies true arithmetic (Theorem 6.4). There might be genuine models of nonstandard arithmetic in this topos (by McCarty's [7], such cannot exist in $\mathcal{E}\!f\!f$: see also [18]). Since Pitts' local operator is induced by the collection of cofinite subsets of $\mathbb{N}$, this is reminiscent of Moerdijk and Palmgren's work on intuitionistic nonstandard models [10,11] obtained by filters.

There are other reasons why one should be interested in the lattice of local operators in $\mathcal{E}\!f\!f$. It is a Heyting algebra in which, as we saw, the Turing degrees embed. It shares this feature with the (dual of the) Medvedev lattice [8], which enjoys a lot of attention these days. Apart from the work by Sorbi and Terwijn (see, e.g., [15,17,16]) who study the logical properties of this lattice, there is the program *Degree Theory*: *a New Beginning* of Steve Simpson, who argues that degree theory should be studied within the Medvedev lattice. From his plenary address 'Mass Problems' at the Logic Colloquium meeting in Bern, 2008 [14]: "*In the 1980s and 1990s, degree theory fell into disrepute. In my opinion, this decline was due to an excessive concentration on methodological aspects, to the exclusion of foundationally significant aspects*". Indeed, it is commonplace in mathematics, in order to study certain structures, to embed them into larger ones with better properties (the passage from ring elements to ideals in number theory; the passage from elements of a structure to types in model theory). By the way, the relationship between the Medvedev lattice and the lattice of local operators in $\mathcal{E}\!f\!f$ seems a worthwhile research project.

This paper is organized as follows. Section 1 reminds the reader of some generalities about the subobject classifier Ω, its set of monotone endomaps and local operators, for as much as is relevant to this paper. Section 2 studies these things in the effective topos. Section 3 recalls known facts from the (limited) literature on the subject. In Section 4 we introduce our main innovation: the concept of sights. Section 5, Calculations, then presents our results. Finally, we present a concrete definition of truth for first-order arithmetic in subtoposes corresponding to local operators, using the language of sights.

0.1. Notation

In this paper, juxtaposition of two terms for numbers: nm will almost always stand for: the result of the n-th partial recursive function to m. The only exception is in the conditions in statements in Section 5, where '$2m$' really means 2 times m, and in the proof of 5.3 where dm also means d times m. We hope the reader can put up with this.

We use the Kleene symbol $\simeq$ between two possibly undefined terms. We use $\langle -,\ldots,-\rangle$ for coded sequences and $(-)_i$ for the i-th element of a coded sequence. The symbol $*$ between coded sequences means: take the code of the concatenated sequence; so if $a=\langle a_0,\ldots,a_{n-1}\rangle$ and $b=\langle b_0,\ldots,b_{m-1}\rangle$ then $a*b=\langle a_0,\ldots,a_{n-1},b_0,\ldots,b_{m-1}\rangle$. We use $\lambda x.t$ for a standard index of a partial recursive function sending x to t.

We employ the logical symbols $\wedge$, $\to$ etc. between formulas, but in the context of $\mathcal{E}\!f\!f$ also between subsets of $\mathbb{N}$, where

$$A\wedge B=\left\{\langle a,b\rangle \,\middle|\, a\in A,\ b\in B\right\}$$

$$A\to B=\{e \mid \text{for all } a\in A,\ ea \text{ is defined and in } B\}$$

For further, unexplained, standard notations regarding the effective topos, we refer to the monograph [21].

1. Subobject classifier, monotone maps and local operators

We shall use the internal language of toposes freely; we refer to one of several available text books on Topos Theory [5,6,4] for expositions of this topic.

If $1\xrightarrow{\text{true}}\Omega$ is a subobject classifier, elements of Ω will act as propositions (Ω is the power set of a one-element set $\{*\}$; and the element p of Ω will also denote the proposition "$*\in p$"); hence Ω is a model of second-order intuitionistic propositional logic. When we use an expression from this logic and say that it 'holds', or is 'true', we have this standard interpretation in mind.

Top and bottom elements of Ω are denoted by $\top$ and $\bot$, respectively.

Definition 1.1. A *local operator* is a map $j:\Omega\to\Omega$ such that the following statements are true:

(a) $\forall p.\ p\to j(p)$,
(b) $\forall pq.\ j(p\wedge q)\leftrightarrow j(p)\wedge j(q)$,
(c) $\forall p.\ j(j(p))\to j(p)$.