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Cardinal characteristics, projective wellorders and large continuum

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ABSTRACT

We extend the work of Fischer et al. (2011) [6] by presenting a method for controlling cardinal characteristics in the presence of a projective wellorder and $2^{\aleph_0} > \aleph_2$. This also answers a question of Harrington (1977) [9] by showing that the existence of a Δ^1_3 wellorder of the reals is consistent with Martin's axiom and $2^{\aleph_0} = \aleph_3$.

1. Introduction

In [6] the present authors established the consistency of the existence of a Π^1_2 maximal almost disjoint family together with a lightface projective wellorder and $\mathfrak{b} = 2^{\aleph_0} = \aleph_3$. As the argument used there was only suitable for handling countable objects, it left open the problem of obtaining projective wellorders with $2^{\aleph_0}$ greater than $\aleph_2$ while simultaneously controlling cardinal characteristics of prominent interest. We solve this problem in the present paper, using an iteration based on the specialization and branching of Suslin trees. As an application we obtain the consistency of $\mathfrak{p} = \mathfrak{b} = \aleph_2 < \mathfrak{a} = \mathfrak{s} = 2^{\aleph_0} = \aleph_3$ with a lightface Δ^1_3 wellorder.

A consequence of our work is the consistency of Martin's axiom with a lightface Δ^1_3 wellorder and $2^{\aleph_0} = \aleph_3$. This improves a result of [8], where $2^{\aleph_0} = \aleph_2$ was obtained, and also answers a question of Harrington from [9], where he obtained the same result with a boldface Δ^1_3 wellorder.

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2. Martin's axiom, projective wellorders and large continuum

We work over the constructible universe L. Fix a canonical sequence $\vec{S} = \langle S_\alpha\colon 1 < \alpha < \omega_3\rangle$ of stationary subsets of $\omega_2 \cap \mathrm{cof}(\omega_1)$ and a nicely definable almost disjoint family $\vec{B} = \langle B_\xi\colon \xi \in \omega_2\rangle$ of subsets of ω_1. More precisely $\vec{S}$ is Σ_1-definable over L_{ω_3} with parameter ω_2 (the S_α's are obtained from a $\diamondsuit$-sequence as in [6]) and $\vec{B}$ is Σ_1-definable over L_{ω_2} with parameter ω_1. For each $\alpha < \omega_3$, let W_α be the L-least subset of ω_2 which codes α. Say that a transitive ZF^- model $\mathcal{M}$ is *suitable* if $\omega_2^{\mathcal{M}}$ exists and $\omega_2^{\mathcal{M}} = \omega_2^{L^{\mathcal{M}}}$. From this it follows, of course, that $\omega_1^{\mathcal{M}} = \omega_1^{L^{\mathcal{M}}}$.

We will define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta\colon \alpha \leqslant \omega_3,\ \beta < \omega_3\rangle$ such that in $L^{\mathbb{P}_{\omega_3}}$, MA holds, $2^\omega = \omega_3$, and there is a Δ^1_3-definable wellorder of the reals. The construction can be thought of as a preliminary stage followed by a coding stage. In the preliminary stage we provide the necessary apparatus, in order to force a Δ^1_3 definition of our wellorder of the reals.

Preliminary stage. For each $0 < \alpha < \omega_3$ and $n \in \omega$, let $\mathbb{K}^0_{\omega\cdot\alpha+n}$ be the poset for adding a Suslin tree $T_{\omega\cdot\alpha+n}$ with countable conditions, see [10, Theorem 15.23]. Let $\mathbb{K}_{0,\alpha} = \prod_{n\in\omega} \mathbb{K}^0_{\omega\cdot\alpha+n}$ with full support. Then $\mathbb{K}_{0,\alpha}$ is countably closed and has size 2^ω. In particular, it does not collapse cardinals provided that CH holds in the ground model.

In what follows we shall identify the T_α's with subsets of ω_1 using the L-least bijection between $\omega^{<\omega_1}$ and ω_1. And vice versa, the phrase "$A \subset \omega_1$ is an ω_1-tree" means throughout the paper that the preimage of A under the L-least bijection between $\omega^{<\omega_1}$ of L and ω_1 is an ω_1-tree. (We can consider such a preimage only in models of $\omega_1 = \omega_1^L$, which is the case in suitable models.)

In $L^{\mathbb{K}_{0,\alpha}}$, code $T_{\omega\cdot\alpha+n}$ via a stationary kill of $S_{\omega_1\cdot(\omega\cdot\alpha+n)+\gamma}$ for $\gamma \in T_{\omega\cdot\alpha+n}$. More precisely, for every $1 \leqslant \alpha < \omega_3$ let $\mathbb{K}_{1,\alpha,n} = \prod_{\gamma\in\omega_1} \mathbb{K}^1_{\alpha,n,\gamma}$ with full support where for $\gamma \in T_{\omega\cdot\alpha+n}$, $\mathbb{K}^1_{\alpha,n,\gamma}$ adds a closed unbounded subset $C_{\omega_1\cdot(\omega\cdot\alpha+n)+\gamma}$ of ω_2 disjoint from $S_{\omega_1\cdot(\omega\cdot\alpha+n)+\gamma}$ and for $\gamma \notin T_{\omega\cdot\alpha+n}$, $\mathbb{K}^1_{\alpha,n,\gamma}$ is the trivial poset. Then $\mathbb{K}_{1,\alpha} = \prod_{n\in\omega} \mathbb{K}_{1,\alpha,n}$ with full support is countably closed, ω_2-distributive, and ω_3-c.c. provided that GCH holds in the ground model.[2]

Next, we shall introduce some auxiliary notation. For a set X of ordinals we denote by $0(X)$, $I(X)$, and $II(X)$ the sets $\{\eta\colon 3\eta \in X\}$, $\{\eta\colon 3\eta+1 \in X\}$ and $\{\eta\colon 3\eta+2 \in X\}$, respectively. Let $Even(X)$ be the set of even ordinals in X and $Odd(X)$ be the set of odd ordinals in X.

In the following we treat 0 as a limit ordinal. Let $D_{\omega\cdot\alpha+n}$ be a subset of ω_2 coding $W_{\omega\cdot\alpha+n}$, $W_{\omega\cdot\alpha}$, and the sequence $\langle C_{\omega_1\cdot(\omega\cdot\alpha+n)+\gamma}\colon \gamma \in T_{\omega\cdot\alpha+n}\rangle$. More precisely, $0(D_{\omega\cdot\alpha+n}) = W_{\omega\cdot\alpha+n}$, $I(D_{\omega\cdot\alpha+n}) = W_{\omega\cdot\alpha}$, and $II(D_{\omega\cdot\alpha+n})$ equals

$$\chi\big(\big\{\langle\gamma,\eta\rangle\colon \gamma \in T_{\omega\cdot\alpha+n},\ \eta \in C_{\omega_1\cdot(\omega\cdot\alpha+n)+\gamma}\big\}\big),$$

where $\chi : \omega_1 \times \omega_2 \to \omega_2$ is some nicely definable bijection. Let $E_{\omega\cdot\alpha+n}$ be the club in ω_2 of intersections with ω_2 of elementary submodels of $L_{(\omega\cdot\alpha+n)+\omega_2}[D_{\omega\cdot\alpha+n}]$ which contain $\omega_1 \cup \{D_{\omega\cdot\alpha+n}\}$ as a subset. (These elementary submodels form an ω_2-chain.) Now choose $Z_{\omega\cdot\alpha+n}$ to be a subset of ω_2 such that $Even(Z_{\omega\cdot\alpha+n}) = D_{\omega\cdot\alpha+n}$, and if $\beta < \omega_2$ is $\omega_2^{\mathcal{M}}$ for some suitable model $\mathcal{M}$ such that $Z_{\omega\cdot\alpha+n} \cap \beta \in \mathcal{M}$, then β belongs to $E_{\omega\cdot\alpha+n} \cap E_{\omega\cdot\alpha}$. (This is easily done by placing in $Z_{\omega\cdot\alpha+n}$ a code for a bijection $\phi : \beta_1 \to \omega_1$ on the interval $(\beta_0, \beta_0+\omega_1)$ for each adjacent pair $\beta_0 < \beta_1$ from $E_{\omega\cdot\alpha+n} \cap E_{\omega\cdot\alpha}$.) Using the same argument as in [6] we have:

$(*)_{\alpha,n}$: If $\beta < \omega_2$ and $\mathcal{M}$ is any suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $Z_{\omega\cdot\alpha+n} \cap \beta$, $Z_{\omega\cdot\alpha} \cap \beta$, $T_{\omega\cdot\alpha+n} \in \mathcal{M}$, then $\mathcal{M} \vDash \psi(\omega_1, \omega_2, Z_{\omega\cdot\alpha+n} \cap \beta, T_{\omega\cdot\alpha+n}, Z_{\omega\cdot\alpha} \cap \beta)$, where $\psi(\omega_1, \omega_2, Z, T, Z')$ is the formula

"$0(Even(Z))$ and $I(Even(Z)) = I(Even(Z'))$ are the L-least codes for ordinals $\omega\cdot\tilde{\alpha}+n$ and $\omega\cdot\tilde{\alpha}$ for some $\tilde{\alpha} \in \omega_3^{\mathcal{M}}$ and $n \in \omega$, respectively, and $\chi^{-1}[II(Even(Z))] = \{\langle\gamma,\eta\rangle\colon \gamma \in T,\ \eta \in \bar{C}_\gamma\}$, where T is an ω_1-tree and $\bar{C}_\gamma$ is a closed unbounded subset of ω_2 disjoint from $S_{\omega_1\cdot(\omega\cdot\tilde{\alpha}+n)+\gamma}$ for all $\gamma \in T$".

In $L^{\mathbb{K}_{0,\alpha} * \mathbb{K}_{1,\alpha}}$ let $\mathbb{K}^2_{\alpha,n}$ add a subset $X_{\omega\cdot\alpha+n}$ of ω_1 which almost disjointly codes $Z_{\omega\cdot\alpha+n}$. More precisely, let $\mathbb{K}^2_{\alpha,n}$ be the poset of all pairs $\langle s, s^*\rangle \in [\omega_1]^{<\omega_1} \times [Z_{\omega\cdot\alpha+n}]^{<\omega_1}$, where a pair $\langle t, t^*\rangle$ extends $\langle s, s^*\rangle$ if and only if t end-extends s and $t\backslash s \cap B_\xi = \emptyset$ for every $\xi \in s^*$. Let $\mathbb{K}_{2,\alpha} = \prod_{n\in\omega} \mathbb{K}^2_{\alpha,n}$ with full support. Then $\mathbb{K}_{2,\alpha}$ is countably closed and ω_2-c.c. provided that CH holds in the ground model.

As a result of this manipulation we get the following:

$(**)_{\alpha,n}$: If $\beta < \omega_2$ and $\mathcal{M}$ is any suitable model such that $\omega_1 \subset \mathcal{M}$, $\omega_2^{\mathcal{M}} = \beta$, and $X_{\omega\cdot\alpha+n}, X_{\omega\cdot\alpha}, T_{\omega\cdot\alpha+n} \in \mathcal{M}$, then $\mathcal{M} \vDash \phi(\omega_1, \omega_2, X_{\omega\cdot\alpha+n}, T_{\omega\cdot\alpha+n}, X_{\omega\cdot\alpha})$, where $\phi(\omega_1, \omega_2, X, T, X')$ is the following formula:

"Using the sequence $\vec{B}$, the sets X, X' almost disjointly code subsets Z, Z' of ω_2 such that $\psi(\omega_1, \omega_2, Z, T, Z')$ holds".

Fix ϕ as above and consider the following poset:

[2] A more general fact will be proven later after we define the final poset.