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Relational dual tableau decision procedures and their applications to modal and intuitionistic logics

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ABSTRACT

We study a class $\mathcal{DL}$ of certain decidable relational logics of binary relations with a single relational constant and restricted composition. The logics in $\mathcal{DL}$ are defined in terms of semantic restrictions on the models. The main contribution of the present article is the construction of relational dual tableau decision procedures for the logics in $\mathcal{DL}$. The systems are constructed in the framework of the original methodology of relational proof systems, determined only by axioms and inference rules, without any external techniques. All necessary bookkeeping is contained in the proof tree itself and used according to the explicit rules. All the systems are deterministic, producing exactly one proof tree for every formula. Furthermore, we show how the systems for logics in $\mathcal{DL}$ can be used as deterministic decision procedures for some modal and intuitionistic logics.

1. Introduction

Relational logics and their dual tableau proof systems have been studied systematically in the last decades (see e.g., [12,2,13,4,1,6,3,9,17,7]). The best known relational logic is the logic RL of binary relations originated in [16]. The formulas of RL are intended to represent statements saying that two objects are related. RL is a logical counterpart to the class RRA of representable relation algebras given by Tarski in [20] (cf. [21]). It is well known that a relational formula xTy is RL-valid if and only if $T = 1$ is true in all representable relation algebras. Therefore, since the equational theory of RRA is undecidable (see [21]), the problem of validity of RL-formulas is also undecidable. Moreover, it is known that RRA is not finitely axiomatizable, as shown by Monk in [14].

The origin of dual tableaux goes back to the paper [18] of Rasiowa and Sikorski, in which the authors presented a cut-free deduction system for the classical first-order logic without equality (see also [19]). Systems in the Rasiowa–Sikorski style are 'top–down' validity checkers, while the well-known tableau systems are unsatisfiability checkers. In [8], it has been proved that the two systems are dual to each other, that is, there is a structure-preserving bijection which transforms a proof in one of those systems into a proof in the other one. For this reason, systems in the Rasiowa–Sikorski style are called

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dual tableaux. It can be also proved that each proof in the Rasiowa–Sikorski style system for first-order logic with equality can be transformed into a proof in the Hilbert-style system, Gentzen sequent calculus, or resolution system. For a detailed overview on the relationships between dual tableaux and Gentzen calculi, see [12] (cf. [17, Ch. 1]).

A very recent comprehensive survey on dual tableaux is the book [17]. It presents the foundations of dual tableaux together with a number of their applications both to logics traditionally dealt with in mathematics and philosophy (modal, intuitionistic, relevant, and many-valued logics) and to various applied theories of computational logic (temporal, spatial, fuzzy, dynamic programming logics).

Relational dual tableau systems are based on the Rasiowa–Sikorski system for first-order logic with equality (see [17, Ch. 1]). The common language of most of relational dual tableaux is the language of RL. The main advantage of the relational methodology is a uniform formalism suitable for representing the three basic components of formal systems: syntax, semantics, and deduction apparatus. Hence, the relational approach can be seen as a general framework for representing, investigating, implementing, and comparing theories with incompatible languages and/or semantics. It has been shown that a great variety of logics can be represented within the relational logic, in particular modal, temporal, spatial, information, program, as well as intuitionistic, and many-valued, among others.

The relational approach enables us to build dual tableaux in a systematic, modular way. The following specific methodological principles of constructing relational dual tableaux make such a broad applicability of these systems possible. First, deduction rules are defined for the common relational core of the theories. These rules constitute a basis of all the relational dual tableau proof systems. Then, given a theory, a truth-preserving translation is defined from the language of the theory into an appropriate language of relations (most often binary). Next, specific rules are added to the basic set of rules. They reflect the semantic constraints assumed in the models of the given theory. As a consequence, we need not implement each deduction system from scratch; it suffices to extend the basic system with a module corresponding to the specific part of a theory under consideration. Furthermore, relational dual tableaux are powerful tools not only for verification of validity but also for proving entailment, model checking (i.e., verification of the truth of a statement in a fixed finite model), and satisfaction (i.e., verification that a statement is satisfied by given elements of a finite model). Examples of relational dual tableaux that realize the above-mentioned logical tasks are presented in [17].

The relational logic RL is undecidable, but there are many fragments of RL which are known to be decidable. However, the general methodology of construction of relational dual tableaux does not guarantee that the constructed system will be a decision procedure. Most often, the dual tableau for RL can be used as a sound and complete proof system for a decidable relational logic. However, since the RL-dual tableau contains the rule for composition, which may be applied infinitely many times, it cannot be used as a decision procedure.

Over the years, dual tableaux have been constructed for a great variety of decidable theories, but relatively little effort has gone into developing their relational decision procedures in dual tableau style. Clearly, the existence of a decision procedure for a decidable logic is very important for practical reasons when we implement a system. The relational dual tableau for the logic RL has been implemented (see e.g., [4]), but its implementations are not effective when used for verification of validity in decidable fragments of RL. As far as we know, relational decision procedures in dual tableau style have been constructed for fragments of the logic RL corresponding to the class of first-order formulas with only universal quantifiers (possibly followed by existential quantifiers) in prenex normal form (see [17]), the relational logic corresponding to the modal logic K (see [10] and [11]), and for some description logics (see [3]).

The aim of this paper is to fill (at least partially) an existing gap in the current state of the art in relational dual tableau decision procedures. We study a class $\mathcal{DL}$ of decidable logics in a relational language with one relational constant and restricted composition. The logics in the class $\mathcal{DL}$ are defined in terms of semantic conditions on the models. The conditions we study are reflexivity, transitivity, and heredity. We construct relational dual tableau decision procedures for the logics in the class $\mathcal{DL}$. All the systems are constructed in the framework of the original methodology of relational proof systems, determined only by axioms and inference rules, without any external techniques. That is, each of the systems for $\mathcal{DL}$ is not only a base for an algorithm verifying validity of a formula of a given logic, but is *itself* a decision procedure, with all the necessary bookkeeping built into the rules. Furthermore, all the dual tableau decision procedures presented in this paper are deterministic. As a consequence, each system generates a unique finite proof tree for a given formula. Hence, different implementations can be tested against each other by checking that they always produce identical trees for the same formula. This is not the case with most of the existing tableau-*based* decision procedures for modal logics, where for complete termination we need to backtrack over the different choices of principal formulas until we find a closed branch. That is, in order to obtain a decision procedure from a tableau system in the general case, the inference rules need to be complemented by a strategy of finding a proof.

Finally, we show that the systems for logics in $\mathcal{DL}$ can be used as relational decision procedures for some modal and intuitionistic logics, in particular for the modal logics K, T, K4, S4, and for the intuitionistic logic INT. In the presentation we follow the notational convention used in the book [17].

The paper is organized as follows. In Section 2, we define a class $\mathcal{DL}$ of decidable relational logics in terms of syntax and semantics. Dual tableaux for the logics in $\mathcal{DL}$ together with a proof of termination are presented in Section 3. Their soundness and completeness are proved in Sections 4 and 5, respectively. In Section 6, we show that the systems for $\mathcal{DL}$ can be applied to some modal and intuitionistic logics as their deterministic decision procedures. Final remarks and prospects for future work are described in Section 7.