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# Many countable support iterations of proper forcings preserve Souslin trees [☆]

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ABSTRACT

We show that many countable support iterations of proper forcings preserve Souslin trees. We establish sufficient conditions in terms of games and we draw connections to other preservation properties. We present a proof of preservation properties in countable support iterations in the so-called Case A that does not need a division into forcings that add reals and those who do not.



## 0. Introduction

This work is related to Juhász' question [14]: "Does Ostaszewski's club principle imply the existence of a Souslin tree?" We recall the club principle (also written $\clubsuit$): There is a sequence $\langle A_\alpha \colon \alpha \text{ a limit ordinal} < \omega_1\rangle$ with the following properties: For every countable limit ordinal $\alpha$, $A_\alpha$ is cofinal in $\alpha$ and for any uncountable $X \subseteq \omega_1$ there are stationarily many $\alpha$ with $A_\alpha \subseteq X$. Such a sequence is called a $\clubsuit$-sequence. The club principle was introduced in [15].

Partial positive answers are known: Let $\mathcal{M}$ denote the ideal of meager sets. In every model of the club principle and $\mathrm{cov}(\mathcal{M}) > \aleph_1$ by Miyamoto [5, Section 4] there are Souslin trees. Brendle showed [5, Theorem 6]:

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In every model of the club principle and $\mathrm{cof}(\mathcal{M}) = \aleph_1$ there are Souslin trees. In this paper we give examples of models satisfying the club principle, the existence of Souslin trees, $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathrm{cof}(\mathcal{M}) = \aleph_2$ (i.e., neither of the sufficient conditions mentioned above holds).

Assume that we start with a ground model satisfying $\diamondsuit_{\omega_1}$ and that we force with a proper countable support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta\colon \alpha \leqslant \kappa,\ \beta < \omega_2\rangle$ of length $\omega_2$. For this scenario in [12] we showed: If the single step forcings are suitable forcings from [16] (with finite or countable $\mathrm{H}(n)$, see Section 2.1), then the final model will satisfy the club principle. Note that the assumption of the diamond in the ground model is actually not necessary, since after $\omega_1$ iteration steps of any forcing with two incompatible conditions with countable support $\diamondsuit_{\omega_1}$ holds anyway [11, Chapter 7, Theorem 8.3] and the length of our iterations is $\omega_2$.

Let us look at the countable support iteration of length $\omega_2$ of Miller forcing: According to the mentioned result, after $\omega_1$ many steps we get $\diamondsuit_{\omega_1}$ and therefore a Souslin tree in the intermediate extension. Theorem 2.1 together with the results in Section 4 show that any countable support iteration of Miller forcing preserves Souslin trees. Hence after $\omega_2$ many iteration steps there is a Souslin tree. Moreover by [12] the club principle holds. It is known that in the Miller model $\mathfrak{d} = \aleph_2$ (and hence $\mathrm{cof}(\mathcal{M}) = \aleph_2$) and $\mathrm{cov}(\mathcal{M}) = \aleph_1$. A countable support iteration of length $\omega_2$ of Blass–Shelah forcing gives another model of $\mathfrak{d} = \aleph_2$ and $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and the club principle. Blass–Shelah forcing is not $\omega$-Cohen preserving (see Definition 3.1) and increases the splitting number (see [3, Proposition 3.1]). Besides these two particular examples, the main technical work in this paper is a study of the preservation of Souslin trees.

We refer the reader to [2] for the definitions of cardinal characteristics, and to [12] for reading about the club principle. For background about properness we refer the reader to [22] and the more detailed introductions in [6,1]. In forcing notions, $q > p$ means that $q$ is stronger than $p$. The paper is organised as follows:

In Section 1 we give some conditions on a forcing in terms of games that imply that the forcing is $(T, Y, \mathcal{S})$-preserving. A special case of $(T, Y, \mathcal{S})$-preserving is preserving the Souslinity of an $\omega_1$-tree.

In Section 2 we show that for some tree-creature forcings from [16] the player COM has a winning strategy in one of the games from Section 1. Hence these forcings preserve Souslin trees. Without the games, we show that some linear creature forcings from [16] are $(T, Y, \mathcal{S})$-preserving. There are non-Cohen preserving examples.

For the wider class of non-elementary proper forcings we show in Section 3 that $\omega$-Cohen preserving for certain candidates implies $(T, Y, \mathcal{S})$-preserving.

In Section 4 we give a less general but hopefully more easily readable presentation of a result from [22, Chapter 18, §3]: If all iterands in a countable support iteration are proper and $(T, Y, \mathcal{S})$-preserving, then also the iteration is $(T, Y, \mathcal{S})$-preserving. This is a presentation of the so-called Case A in which a division in forcings that add reals and those who do not is not needed.

## 1. A sufficient condition for $(T, Y, \mathcal{S})$-preserving

We introduce two games $\partial^\iota(\mathbb{P}, p)$, $\iota = 1, 2$, that are games about the completeness of the notion of forcing $\mathbb{P}$ above $p$. Similar games appear in [17,19,18]. We let $\mathbf{G}_{\mathbb{P}} = \{(\check{p}, p)\colon p \in \mathbb{P}\}$ be the standard name for a $\mathbb{P}$-generic filter. If it is clear which $\mathbb{P}$ is meant we write just $\underset{\sim}{\mathbf{G}}$.

**Definition 1.1.** Let $\mathbb{P}$ be a notion of forcing and $p \in \mathbb{P}$. We define the games $\partial^\iota(\mathbb{P}, p)$, $\iota = 1, 2$. The moves look the same for both games, and only in the winning conditions they are different.

(1) The game $\partial^1(\mathbb{P}, p)$ is played in $\omega$ rounds. In round $n$, player COM chooses an $\ell_n \in \omega \setminus \{0\}$ and a sequence $\langle p_{n,\ell}\colon \ell < \ell_n\rangle$ of conditions $p_{n,\ell} \in \mathbb{P}$ and then player INC plays $\langle q_{n,\ell}\colon n < \ell_n\rangle$ such that $p_{n,\ell} \leqslant q_{n,\ell}$. After $\omega$ rounds, COM wins the game iff there is $q \geqslant p$ such that for each $n$,

$$\{q_{n,\ell}\colon \ell < \ell_n\} \text{ is predense above } q.$$