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Squares and covering matrices

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ABSTRACT

Viale introduced covering matrices in his proof that SCH follows from PFA. In the course of the proof and subsequent work with Sharon, he isolated two reflection principles, CP and S, which, under certain circumstances, are satisfied by all covering matrices of a certain shape. Using square sequences, we construct covering matrices for which CP and S fail. This leads naturally to an investigation of square principles intermediate between $\square_\kappa$ and $\square(\kappa^+)$ for a regular cardinal κ. We provide a detailed picture of the implications between these square principles.

1. Introduction

There is a fundamental and well-studied tension in set theory between large cardinals and reflection phenomena on the one hand and combinatorial principles (various square principles, in particular) witnessing incompactness (see, for example, [2]) on the other. Reflection and large cardinals place limits on the type of combinatorial structures which can exist, and vice versa.

Covering matrices were introduced by Viale in his proof that the Singular Cardinals Hypothesis follows from the Proper Forcing Axiom [11]. Here and in later work with Sharon [9], Viale also isolated two natural properties, $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$, which can hold for a given covering matrix $\mathcal{D}$. The statement that $\mathrm{CP}(\mathcal{D})$ (or $\mathrm{S}(\mathcal{D})$) holds for every covering matrix $\mathcal{D}$ of a certain type can be seen as a reflection statement and is thus at odds with the aforementioned incompactness phenomena.

We start this paper by constructing various covering matrices for which $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ fail and investigating the relationship between the failure of $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ and the existence of square sequences. This

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leads naturally to the definition of certain square principles which, for a regular, uncountable cardinal κ, are intermediate between $\square_\kappa$ and $\square(\kappa^+)$. We conclude by obtaining a detailed picture of the implications between these square principles.

Our notation is for the most part standard. Unless otherwise specified, the reference for all notation and definitions is [6]. If A is a set and θ is a cardinal, then $[A]^\theta$ is the collection of subsets of A of size θ. If A is a set of ordinals and $\alpha < \sup(A)$ is an ordinal of uncountable cofinality, we say A *reflects* at α if $A \cap \alpha$ is stationary in α. If A is a set of ordinals, then A' denotes the set of limit ordinals of A, i.e. the set of α such that $A \cap \alpha$ is unbounded in α, and $\mathrm{otp}(A)$ denotes the order type of A. If $\phi : A \to B$ is a (partial) function and $X \subseteq A$, then $\phi[X]$ is the image of X under ϕ, and $\phi \restriction X$ is the restriction of ϕ to $\mathrm{dom}(\phi) \cap X$. If λ is a cardinal and $\mu < \lambda$ is a regular cardinal, then $S^\lambda_\mu = \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \mu\}$. $S^\lambda_{<\mu}$ is defined in the obvious way. If s is a sequence, then $|s|$ denotes the length of s, and, if t is also a sequence, $s^\frown t$ denotes the concatenation of the two.

2. Covering matrices

Definition. Let $\theta < \lambda$ be regular cardinals. $\mathcal{D} = \{D(i,\beta) \mid i < \theta, \beta < \lambda\}$ is a *θ-covering matrix for λ* if:

1. For all $\beta < \lambda$, $\beta = \bigcup_{i<\theta} D(i,\beta)$.
2. For all $\beta < \lambda$ and all $i < j < \theta$, $D(i,\beta) \subseteq D(j,\beta)$.
3. For all $\beta < \gamma < \lambda$ and all $i < \theta$, there is $j < \theta$ such that $D(i,\beta) \subseteq D(j,\gamma)$.

$\beta_{\mathcal{D}}$ is the least β such that for all $\gamma < \lambda$ and all $i < \theta$, $\mathrm{otp}(D(i,\gamma)) < \beta$. $\mathcal{D}$ is *normal* if $\beta_{\mathcal{D}} < \lambda$.

$\mathcal{D}$ is *transitive* if, for all $\alpha < \beta < \lambda$ and all $i < \theta$, if $\alpha \in D(i,\beta)$, then $D(i,\alpha) \subseteq D(i,\beta)$.

$\mathcal{D}$ is *uniform* if for all $\beta < \lambda$ there is $i < \theta$ such that $D(j,\beta)$ contains a club in β for all $j \geqslant i$. (Note that this is equivalent to the statement that there is $i < \theta$ such that $D(i,\beta)$ contains a club in β.)

$\mathcal{D}$ is *closed* if for all $\beta < \lambda$, all $i < \theta$, and all $X \in [D(i,\beta)]^{\leqslant\theta}$, $\sup X \in D(i,\beta)$.

The first part of this paper will be concerned with constructing covering matrices for which the following two reflection properties fail.

Definition. Let $\theta < \lambda$ be regular cardinals, and let $\mathcal{D}$ be a θ-covering matrix for λ.

1. $\mathrm{CP}(\mathcal{D})$ holds if there is an unbounded $T \subseteq \lambda$ such that for every $X \in [T]^\theta$, there are $i < \theta$ and $\beta < \lambda$ such that $X \subseteq D(i,\beta)$ (in this case, we say that $\mathcal{D}$ *covers* $[T]^\theta$).
2. $\mathrm{S}(\mathcal{D})$ holds if there is a stationary $S \subseteq \lambda$ such that for every family $\{S_j \mid j < \theta\}$ of stationary subsets of S, there are $i < \theta$ and $\beta < \lambda$ such that, for every $j < \theta$, $S_j \cap D(i,\beta) \neq \emptyset$.

Definition. Let $\theta < \lambda$ be regular cardinals. $\mathrm{R}(\lambda,\theta)$ is the statement that there is a stationary $S \subseteq \lambda$ such that for every family $\{S_j \mid j < \theta\}$ of stationary subsets of S, there is $\alpha < \lambda$ of uncountable cofinality such that, for all $j < \theta$, S_j reflects at α.

If $\mathcal{D}$ is a nice enough covering matrix, then $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ are equivalent and $\mathrm{R}(\lambda,\theta)$ implies both. The following is proved in [9]:

Lemma 1. *Let $\theta < \lambda$ be regular cardinals, and let $\mathcal{D}$ be a θ-covering matrix for λ.*

1. *If $\mathcal{D}$ is transitive, then $\mathrm{S}(\mathcal{D})$ implies $\mathrm{CP}(\mathcal{D})$.*
2. *If $\mathcal{D}$ is closed, then $\mathrm{CP}(\mathcal{D})$ implies $\mathrm{S}(\mathcal{D})$.*
3. *If $\mathcal{D}$ is uniform, then $\mathrm{R}(\lambda,\theta)$ implies $\mathrm{S}(\mathcal{D})$.*