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Dichotomy theorems for families of non-cofinal essential complexity ☆

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ABSTRACT

We prove that for every Borel equivalence relation E, either E is Borel reducible to $\mathbb{E}_0$, or the family of Borel equivalence relations incompatible with E has cofinal essential complexity. It follows that if F is a Borel equivalence relation and $\mathscr{F}$ is a family of Borel equivalence relations of non-cofinal essential complexity which together satisfy the dichotomy that for every Borel equivalence relation E, either $E \in \mathscr{F}$ or F is Borel reducible to E, then $\mathscr{F}$ consists solely of smooth equivalence relations, thus the dichotomy is equivalent to a known theorem.

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0. Introduction

A *reduction* of an equivalence relation E on a set X to an equivalence relation F on a set Y is a function $\pi: X \to Y$ with the property that $\forall x_1, x_2 \in X$ $(x_1 \ E \ x_2 \iff \pi(x_1) \ F \ \pi(x_2))$. A topological space is *Polish* if it is second countable and completely metrizable, a subset of such a space is *Borel* if it is in the σ-algebra generated by the underlying topology, and a function between such spaces is *Borel* if pre-images of open sets are Borel. Over the last few decades, the study of Borel reducibility of Borel equivalence relations on Polish spaces has emerged as a central theme in descriptive set theory.

The early development of this area was dominated by dichotomy theorems. There are several trivial ones, such as the fact that if n is a natural number, then for every Borel equivalence relation E on a Polish space, either E is Borel reducible to equality on n, or equality on $n + 1$ is Borel reducible to E. Similarly, either there is a natural number n for which E is Borel reducible to equality on n, or equality on $\mathbb{N}$ is Borel reducible to E.

There are also non-trivial results of this form. By [9], either E is Borel reducible to equality on $\mathbb{N}$, or equality on $2^{\mathbb{N}}$ is Borel reducible to E. And by [5, Theorem 1.1], either E is Borel reducible to equality on $2^{\mathbb{N}}$, or $\mathbb{E}_0$ is Borel reducible to E, where $\mathbb{E}_0$ is the relation on $2^{\mathbb{N}}$ given by $x \ \mathbb{E}_0 \ y \iff \exists n \in \mathbb{N} \forall m \geq n \ x(m) = y(m)$.

Whereas the results we have mentioned thus far concern the global structure of the Borel reducibility hierarchy, [8, Theorem 1] yields a local dichotomy of this form. Namely, that for every Borel equivalence relation E on a Polish space which is Borel reducible to $\mathbb{E}_1$, either E is Borel reducible to $\mathbb{E}_0$, or $\mathbb{E}_1$ is Borel reducible to E, where $\mathbb{E}_1$ is the relation on $(2^{\mathbb{N}})^{\mathbb{N}}$ given by $x \ \mathbb{E}_1 \ y \iff \exists n \in \mathbb{N} \forall m \geq n \ x(m) = y(m)$.

At first glance, one might hope the assumption that E is Borel reducible to $\mathbb{E}_1$ could be eliminated, thereby yielding a new global dichotomy theorem. Unfortunately, [8, Theorem 2] ensures that if E is not Borel reducible to $\mathbb{E}_0$, then there is a Borel equivalence relation with which it is incomparable under Borel reducibility. It follows that only the pairs (F, F') discussed thus far (up to Borel bi-reducibility) satisfy both (1) there is a Borel reduction of F to F' but not vice versa, and (2) for every Borel equivalence relation E on a Polish space, either E is Borel reducible to F, or F' is Borel reducible to E.

As the latter result rules out further global dichotomies of the sort discussed thus far, it is interesting to note that its proof hinges upon the previously mentioned local dichotomy, as well as Harrington's unpublished theorem that the family of orbit equivalence relations induced by Borel actions of Polish groups on Polish spaces is unbounded in the Borel reducibility hierarchy. Here we utilize strengthenings of these results to provide a substantially stronger anti-dichotomy theorem.

Given a property P of Borel equivalence relations, we say that a Borel equivalence relation is *essentially* P if it is Borel reducible to a Borel equivalence relation on a Polish space with the given property. A *Wadge reduction* of a set $A \subseteq X$ to a set $B \subseteq Y$ is a continuous function $\pi: X \to Y$ such that $\forall x \in X$ $(x \in A \iff \pi(x) \in B)$. We say that a Borel equivalence relation E has *essential complexity at least the complexity* of a set $B \subseteq 2^{\mathbb{N}}$ if B is Wadge reducible to every Borel equivalence relation to which E is Borel