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Minimal ambient nuclear C*-algebras

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ABSTRACT

We provide examples of ambient nuclear C*-algebras of non-nuclear C*-algebras with no proper intermediate C*-algebras. In particular this gives the first examples of minimal ambient nuclear C*-algebras of non-nuclear C*-algebras. For this purpose, we study generic Cantor systems of infinite free product groups.

1. Introduction

In 1979, Choi [4] constructed the first example of an ambient nuclear C*-algebra of a non-nuclear C*-algebra. In the celebrated paper of Kirchberg–Phillips [11], they show that any separable exact C*-algebra in fact has an ambient nuclear C*-algebra. (In fact, one can choose it to be isomorphic to the Cuntz algebra $\mathcal{O}_2$.) When we consider reduced group C*-algebras, thanks to Ozawa's result [14], we have more natural ambient

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nuclear C*-algebras, namely, the reduced crossed products of amenable dynamical systems. Ambient nuclear C*-algebras play important roles in the theory of both C*- and von Neumann algebras. We refer the reader to the books [3] and [21] for details. In this paper, we investigate how an ambient nuclear C*-algebra of a non-nuclear C*-algebra can be tight. See the paper [15] for a related topic. Based on (new) results on topological dynamical systems, we give the first example of a minimal ambient nuclear C*-algebra of a non-nuclear C*-algebra. In fact, we have a stronger result: our examples of minimal ambient nuclear C*-algebras have no proper intermediate C*-algebras.

Note that as shown in [23], in contrast to injectivity of von Neumann algebras, nuclearity of C*-algebras is not preserved under taking the decreasing intersection. We also note that the increasing union of non-nuclear C*-algebras can be nuclear. See Remark 3.10 for details. Thus there is no obvious way to provide a minimal ambient nuclear C*-algebra. We also remark that in the von Neumann algebra case, thanks to the bicommutant theorem, for any von Neumann algebra, finding a minimal ambient injective von Neumann algebra is equivalent to finding a maximal injective von Neumann subalgebra. The latter problem draws many researchers's interest. In [19], Popa provided the first concrete examples of maximal injective von Neumann subalgebras. For recent progresses on this problem, see [2] and [8] for instance.

In 1975, Powers [20] invented a celebrated method to study structures of the reduced group C*-algebras. His idea has been applied to more general situations, particularly for reduced crossed products, and to more general groups, by many hands. See [5] for instance. We combine his technique with certain properties of dynamical systems to obtain the following main theorem of the paper.

We say that a group is an infinite free product group if it is a free product of infinitely many nontrivial groups. Throughout the paper, groups are assumed to be countable.

Main Theorem *(Corollary 2.4, Theorem 3.9). Let* Γ *be an infinite free product group with the AP (approximation property) [6] (or equivalently, each free product component has the AP). Then there is an amenable action of* Γ *on the Cantor set* X *with the following property. There is no proper intermediate* C*-*algebra of the inclusion* $\mathrm{C}^*_r(\Gamma) \subset C(X) \rtimes_r \Gamma$. *In particular* $C(X) \rtimes_r \Gamma$ *is a minimal ambient nuclear* C*-*algebra of the non-nuclear* C*-*algebra* $\mathrm{C}^*_r(\Gamma)$.

We remark that it is not known if there is an ambient injective von Neumann algebra (or equivalently, injective von Neumann subalgebra) of a non-injective von Neumann algebra with no proper intermediate von Neumann algebra. We refer the reader to [3] for basic knowledge on C*-algebras and discrete groups. Here we remark that the AP implies exactness (see Section 12.4 of [3]), while the converse is not true [13]. In Main Theorem, we need the AP to determine when a given element of the reduced crossed product sits in the reduced group C*-algebra. (Cf. [23,26].)

In the theory of both measurable and topological dynamical systems, the Baire category theorem is a powerful tool to produce an example with a nice property. For further