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Volume inequalities of convex bodies from cosine transforms on Grassmann manifolds ☆

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The L_p cosine transform on Grassmann manifolds naturally induces finite dimensional Banach norms whose unit balls are origin-symmetric convex bodies in $\mathbb{R}^n$. Reverse isoperimetric type volume inequalities for these bodies are established, which extend results from the sphere to Grassmann manifolds.

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1. Introduction

The solution to the classical isoperimetric problem says that among all convex bodies of given surface area in the Euclidean space $\mathbb{R}^n$, only the ball has maximal volume. It is usually written as the following isoperimetric inequality,

$$S(K) \geq n\omega_n^{\frac{1}{n}} V(K)^{\frac{n-1}{n}},$$

with equality if and only if the convex body K is a ball, where $S(K)$ and $V(K)$ denote the surface area and volume of K, respectively, and ω_n is the volume of the Euclidean unit ball. A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior.

Note that the volume of a convex body can be arbitrarily small when its surface area is fixed. Thus, the isoperimetric inequality can not be reversed with a different constant factor. Establishing a reverse isoperimetric inequality that characterizes cubes, simplices, or other non-spherical convex bodies is a highly interesting problem in convex geometry.

The celebrated work of Keith Ball [2,3] is a landmark in the study of the reverse isoperimetric problem. He proved that for any symmetric convex body K in $\mathbb{R}^n$ there is a volume preserving linear transformation ψ so that the surface area of ψK is no larger than that of a cube of the same volume. If symmetry is not assumed, he proved the similar remarkable result for simplices. In Ball's works [2–4], the notion of isotropy of measures on the unit sphere (see (2.8)) and the Brascamp–Lieb inequality played critical roles. By using the method of mass transportation, Barthe [7] found a new proof of the Brascamp–Lieb inequality and established the reverse Brascamp–Lieb inequality. He then used the inequalities to show new reverse isoperimetric inequalities, and also showed the uniqueness of equality cases of his and Ball's reverse isoperimetric inequalities, see [5–8]. The remarkable work of Ball and Barthe has motivated a series of new studies, see for example, [1,10–13,18–21,24–26,31–33,41–45,56,57]. Some of Ball and Barthe's results were generalized in [42,44] from discrete to arbitrary isotropic measures on the unit sphere and from polytopes to arbitrary convex bodies in $\mathbb{R}^n$. Most recently, Schuster and Weberndorfer [60] proved important reverse isoperimetric inequalities for Wulff shapes of arbitrary isotropic measures and L_2 functions on the unit sphere which further generalize and unify the results of Ball [2,3], Barthe [7], and [44,45].

The purpose of this paper is to extend volume inequalities arising from measures on the unit sphere to volume inequalities for measures on Grassmann manifolds. We study the L_p cosine transforms on Grassmann manifolds which include the spherical cosine and sine transforms as special cases. Reverse isoperimetric inequalities are established for convex bodies that are naturally associated with cosine transforms on Grassmann manifolds, which generalize and unify the results on the unit sphere. New concepts and techniques are introduced for proving these results.

A tool in harmonic analysis that is useful for the reverse isoperimetric problem is the spherical cosine transform. The L_p cosine transform C_p on the unit sphere S^{n-1} gives a natural analytical operator in convex geometric analysis. It induces n-dimensional norms