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New pathways and connections in number theory and analysis motivated by two incorrect claims of Ramanujan

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In Memory of W. Keith Moore, Professor of Mathematics at Albion College

Dedicated to Pratibha Kulkarni who first showed me the beauty of Mathematics

ABSTRACT

The focus of this paper commences with an examination of three (not obviously related) pages in Ramanujan's lost notebook, pages 336, 335, and 332, in decreasing order of attention. On page 336, Ramanujan proposes two identities, but the formulas are wrong – each is vitiated by divergent series. We concentrate on only one of the two incorrect "identities," which may have been devised to attack the extended divisor problem. We prove here a corrected version of Ramanujan's claim, which contains the convergent series appearing in it. The convergent series in Ramanujan's faulty claim is similar to one used by G.F. Voronoï, G.H. Hardy, and others in their study of the classical Dirichlet divisor problem. This now brings us to page 335, which comprises two

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formulas featuring doubly infinite series of Bessel functions, the first being conjoined with the classical circle problem initiated by Gauss, and the second being associated with the Dirichlet divisor problem. The first and fourth authors, along with Sun Kim, have written several papers providing proofs of these two difficult formulas in different interpretations. In this monograph, we return to these two formulas and examine them in more general settings.

The aforementioned convergent series in Ramanujan's "identity" is also similar to one that appears in a curious identity found in Chapter 15 in Ramanujan's second notebook, written in a more elegant, equivalent formulation on page 332 in the lost notebook. This formula may be regarded as a formula for $\zeta(\frac{1}{2})$, and in 1925, S. Wigert obtained a generalization giving a formula for $\zeta(\frac{1}{k})$ for any even integer $k \geq 2$. We extend the work of Ramanujan and Wigert in this paper.

The Voronoï summation formula appears prominently in our study. In particular, we generalize work of J.R. Wilton and derive an analogue involving the sum of divisors function $\sigma_s(n)$.

The modified Bessel functions $K_s(x)$ arise in several contexts, as do Lommel functions. We establish here new series and integral identities involving modified Bessel functions and modified Lommel functions. Among other results, we establish a modular transformation for an infinite series involving $\sigma_s(n)$ and modified Lommel functions. We also discuss certain obscure related work of N.S. Koshliakov. We define and discuss two new related classes of integral transforms, which we call Koshliakov transforms, because he first found elegant special cases of each.

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