Advances in Mathematics 304 (2017) 1055–1079

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Embedding Riemannian manifolds by the heat kernel of the connection Laplacian

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A R T I C L E I N F O

*Article history:*
Received 27 April 2013
Received in revised form 26 October 2015
Accepted 1 May 2016
Available online 22 September 2016
Communicated by Charles Fefferman

*Keywords:*
Vector diffusion map
Spectral geometry
Precompactness
Graph connection Laplacian
Spectral graph theory

A B S T R A C T

Given a class of closed Riemannian manifolds with prescribed geometric conditions, we introduce an embedding of the manifolds into $\ell^2$ based on the heat kernel of the Connection Laplacian associated with the Levi-Civita connection on the tangent bundle. As a result, we can construct a distance in this class which leads to a pre-compactness theorem on the class under consideration.



## 1. Introduction

In [6], the following class of closed Riemannian manifolds $\mathcal{M}_{d,k,D}$ with prescribed geometric constrains is considered:

$$\mathcal{M}_{d,k,D} = \{(M,g)|\,\dim(M) = d,\, \mathrm{Ric}(g) \geq (d-1)kg,\, \mathrm{diam}(M) \leq D\}, \tag{1}$$

where Ric is the Ricci curvature and diam is the diameter. The authors embed $M \in \mathcal{M}_{d,k,D}$ into the space $\ell^2$ of real-valued, square integrable series by considering the heat

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http://dx.doi.org/10.1016/j.aim.2016.05.023

kernel of the Laplace–Beltrami operator of $M$. A distance on $\mathcal{M}_{d,k,D}$, referred to as the *spectral distance* (SD), is then introduced based on the embedding so that the set of isometric classes in $\mathcal{M}_{d,k,D}$ is precompact. Over the past decades many works in the manifold learning field benefit from this embedding scheme, for example, the eigenmaps [3], diffusion maps (DM) [10] and the manifold parameterizations [17,18]. An important practical and theoretical question about how many eigenfunctions one needs to embed the manifold via DM has been answered in [2,21].

Recently, a new mathematical framework, referred to as *vector diffusion maps* (VDM), for organizing and analyzing massive high dimensional data sets, images and shapes was introduced in [23]. In brief, VDM is a mathematical generalization of DM, which numerically is based on a generalization of graph Laplacian, referred to as *graph connection Laplacian* (GCL). While DM are based on the heat kernel of the Laplace–Beltrami operator over the manifold, VDM is based on the heat kernel of the connection Laplacian associated with the Levi-Civita connection on the tangent bundle of the manifold. The introduction of VDM was motivated by the problem of finding an efficient way to organize complex data sets, embed them in a low dimensional space, and interpolate and regress vector fields over the data. In particular, it equips the data with a metric, which we refer to as the *vector diffusion distance* (VDD). VDM has been applied to several different problems, ranging from the cryo-electron microscopy problem [16,25,27] to graph realization problems [11,12] and modern light source imaging technique [1,19]. The application of VDM to the cryo-electron microscopy problem, which is aimed to reconstruct the three dimensional geometric structure of the macromolecule, provides a better organization of the given noisy projection images, and hence a better reconstruction result [16,25,27]. In addition, the GCL can be slightly modified to determine the orientability of a manifold and obtain its orientable double covering if the manifold is non-orientable [22].

In this paper, we consider the same class of closed Riemannian manifolds $\mathcal{M}_{d,k,D}$ and focus on the connection Laplacian associated with the Levi-Civita connection on the tangent bundle. We analyze how the VDM embeds the manifold $M \in \mathcal{M}_{d,k,D}$ into $\ell^2$ based on the heat kernel of the connection Laplacian of the tangent bundle. Based on the VDM and VDD, we introduce a new distance on $\mathcal{M}_{d,k,D}$, referred to as *vector spectral distance* (VSD), which leads to the pre-compactness of the set $\mathcal{M}_{d,k,D}$.

### *1.1. Summary of the work and organization of the paper*

The results of this paper are organized in the following way. In Section 2 the background material and notations are provided.

In Section 3, the VDM and VDD are defined and we discuss the embedding property of VDM in Theorem 3.1 and the local geodesic estimation property of VDD in Theorem 3.2. Indeed, VDM is a diffeomorphic embedding of a manifold $M \in \mathcal{M}_{d,k,D}$ into the Hilbert space $\ell^2$, and if $x, y \in M$ are close enough in the sense of geodesic distance, then the VDD of these two points are closely related to the geodesic distance.