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Homology of left non-degenerate set-theoretic solutions to the Yang–Baxter equation ☆

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A B S T R A C T

This paper deals with left non-degenerate set-theoretic solutions to the Yang–Baxter equation (= LND solutions), a vast class of algebraic structures encompassing groups, racks, and cycle sets. To each such solution there is associated a shelf (i.e., a self-distributive structure) which captures its major properties. We consider two (co)homology theories for LND solutions, one of which was previously known, in a reduced form, for biracks only. An explicit isomorphism between these theories is described. For groups and racks we recover their classical (co)homology, whereas for cycle sets we get new constructions. For a certain type of LND solutions, including quandles and non-degenerate cycle sets, the (co)homologies split into the degenerate and the normalized parts. We express 2-cocycles of our theories in terms of group cohomology, and, in the case of cycle sets, establish connexions with extensions. This leads to a construction of cycle sets with interesting properties.

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Braided homology
Extension
Cubical homology

Contents

1. Introduction

The *Yang–Baxter equation* (= *YBE*) plays a fundamental role in such apparently distant fields as statistical mechanics, particle physics, quantum field theory, quantum group theory, and low-dimensional topology; see for instance [32] for a brief introduction. The study of its solutions has been a vivid research area for the last half of a century. Following Drinfel'd [11], set-theoretic solutions, or *braided sets*, received special attention. Concretely, these are sets X endowed with a *braiding*, i.e., a not necessarily invertible map $\sigma\colon X^{\times 2} \to X^{\times 2}$, often written as $\sigma(a,b) = ({}^{a}b, a^{b})$, satisfying the YBE

$$(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id}) = (\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)\colon X^{\times 3} \to X^{\times 3}. \tag{1.1}$$

Two families of braided sets are particularly well explored:

- The map

  $$\sigma(a,b) = (b, a^{b})$$

  is a braiding if and only if the operation $a \triangleleft b := a^{b}$ is *self-distributive*, in the sense of

  $$(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c). \tag{1.2}$$

  Such datum $(X, \triangleleft)$ is called a *shelf*. The term *rack* is used if moreover the right translations $a \mapsto a \triangleleft b$ are bijections on X for all $b \in X$, which is equivalent to the invertibility of σ. A *quandle* is a rack satisfying $a \triangleleft a = a$ for all a, which