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The diffractive wave trace on manifolds with conic singularities

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ABSTRACT

Let (X, g) be a compact manifold with conic singularities. Taking Δ_g to be the Friedrichs extension of the Laplace–Beltrami operator, we examine the singularities of the trace of the half-wave group $e^{-it\sqrt{\Delta_g}}$ arising from strictly diffractive closed geodesics. Under a generic nonconjugacy assumption, we compute the principal amplitude of these singularities in terms of invariants associated to the geodesic and data from the cone point. This generalizes the classical theorem of Duistermaat–Guillemin on smooth manifolds and a theorem of Hillairet on flat surfaces with cone points.

0. Introduction

In this paper, we consider the trace of the half-wave group $\mathcal{U}(t) \stackrel{\text{def}}{=} e^{-it\sqrt{\Delta_g}}$ on a compact manifold with conic singularities (X, g). Our main result is a description of the singularities of this trace at the lengths of closed geodesics undergoing diffractive interaction with the cone points. Under the generic assumption that

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the cone points of X are pairwise nonconjugate along the geodesic flow, (0.1)

the resulting singularity at such a length $t = L$ has the oscillatory integral representation

$$\int_{\mathbb{R}_\xi} e^{-i(t-L)\cdot\xi}\, a(t,\xi)\, d\xi,$$

where the amplitude a is to leading order

$$a(t,\xi) \sim L_0 \cdot (2\pi)^{\frac{kn}{2}}\, e^{\frac{ik\pi(n-3)}{4}} \cdot \chi(\xi)\, \xi^{-\frac{k(n-1)}{2}} \times \left[\prod_{j=1}^{k} i^{-m_{\gamma_j}} \cdot \boldsymbol{D}_{\alpha_j}(q_j, q_j') \cdot \operatorname{dist}_g^{\gamma_j}(Y_{\alpha_{j+1}}, Y_{\alpha_j})^{-\frac{n-1}{2}} \cdot \Theta^{-\frac{1}{2}}(Y_{\alpha_j} \to Y_{\alpha_{j+1}}) \right]$$

as $|\xi| \to \infty$ and the index j is cyclic in $\{1, \ldots, k\}$. Here, n is the dimension of X and k the number of diffractions along the geodesic, and χ is a smooth function supported in $[1, \infty)$ and equal to 1 on $[2, \infty)$. L_0 is the primitive length, in case the geodesic is an iterate of a shorter closed geodesic. The product is over the diffractions undergone by the geodesic, with $\boldsymbol{D}_{\alpha_j}$ a quantity determined by the functional calculus of the Laplacian on the link of the j-th cone point Y_{α_j}, the factor $\Theta^{-\frac{1}{2}}(Y_{\alpha_j} \to Y_{\alpha_{j+1}})$ is (at least on a formal level) the determinant of the differential of the flow between the j-th and $(j+1)$-st cone points, and m_{γ_j} is the Morse index of the geodesic segment γ_j from the j-th to $(j+1)$-st cone points. All of these factors are described in more detail below.

To give this result some context, we recall the known results for the Laplace–Beltrami operator $\Delta_g = d_g^* \circ d$ on a *smooth* ($\mathcal{C}^\infty$) compact Riemannian manifold (X, g). In this setting, there is a countable orthonormal basis for $L^2(X)$ comprised of eigenfunctions φ_j of Δ_g with eigenvalues $\left\{\lambda_j^2\right\}_{j=0}^{\infty}$ of finite multiplicity and tending to infinity. The celebrated trace formula of Duistermaat and Guillemin [10], a generalization of the Poisson summation formula to this setting, establishes that the quantity

$$\sum_{j=0}^{\infty} e^{-it\lambda_j}$$

is a well-defined distribution on $\mathbb{R}_t$. Moreover, it satisfies the "Poisson relation": it is singular only on the *length spectrum* of (X, g),

$$\operatorname{LSp}(X, g) \stackrel{\text{def}}{=} \{0\} \cup \{\pm L \in \mathbb{R} : L \text{ is the length of a closed geodesic in } (X, g)\}.$$

(This was shown independently by Chazarain [5]; see also [8].) Subject to a nondegeneracy hypothesis, the singularity at the length $t = \pm L$ of a closed geodesic has a specific leading form encoding the geometry of that geodesic—the formula involves the linearized Poincaré map and the Morse index of the geodesic. The proofs of these statements center around the identification