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The rigidity theorems of self shrinkers via Gauss maps ☆

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ABSTRACT

We study the rigidity results for self-shrinkers in Euclidean space by restriction of the image under the Gauss map. The geometric properties of the target manifolds carry into effect. In the self-shrinking hypersurface situation Theorem 3.1 and Theorem 3.2 not only improve the previous results, but also are optimal. In higher codimensional case, using geometric properties of the Grassmannian manifolds (the target manifolds of the Gauss map) we give a rigidity theorem for self-shrinking graphs.

1. Introduction

Minimal submanifolds and self-shrinkers both are special solutions to the mean curvature flow. Those two subjects share many geometric properties, as shown in [3]. We continue to study rigidity properties of self-shrinkers. In the previous work we discussed the gap phenomena for squared norm of the second fundamental form for self-shrinkers [6].

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For submanifolds in Euclidean space we have the important Gauss map, which plays an essential role in the submanifold theory. In the present paper we shall study the gap phenomena of the image under the Gauss maps for self-shrinkers. In the literature [8,3] the polynomial volume growth is an adequate assumption for the complete non-compact self-shrinkers. Ding–Xin [5] showed that the properness shall guarantee the Euclidean volume growth. Afterwards, Cheng–Zhou [2] proved that the inverse is also true. Now, we only study properly immersed self-shrinkers. We pursue the results that a complete properly immersed self-shrinker would become an affine linear subspace or a cylinder, if its Gauss image is sufficiently restricted.

In the next section we show that the Gauss map of a self-shrinker is a weighted harmonic map, which is a conclusion of the Ruh–Vilms type result for self-shrinkers, see Theorem 2.1. We also derive a composition formula for the drift-Laplacian operator defined on self-shrinkers, which enables us to obtain some results of self-shrinkers via properties of the target manifold of the Gauss map in the subsequent sections of this paper.

In §3 we study the codimension one case. If M is an entire graphic self shrinking hypersurface in $\mathbb{R}^{n+1}$, Ecker–Huisken [8] showed that M is a hyperplane under the assumption of polynomial volume growth, which was removed by Wang [16]. Namely, any entire graphic self-shrinking hypersurface in Euclidean space has to be a hyperplane. It is in sharp contrast to the case of minimal graphic hypersurfaces. For constant mean curvature surfaces in $\mathbb{R}^3$ there is a well-known result, due to Hoffman–Osserman–Schoen [10], which shows that a plane or a circular cylinder could be characterized by its Gauss image among other complete constant mean curvature surfaces in $\mathbb{R}^3$. In this circumstance we consider a properly immersed self-shrinking hypersurface M in $\mathbb{R}^{n+1}$. Now the target manifolds of the Gauss map for a self shrinking hypersurface in $\mathbb{R}^{n+1}$ is the unit sphere. We obtain a counterpart of their result and prove that if the image under the Gauss map is contained in an open hemisphere (which includes the case of graphic self shrinking hypersurfaces in $\mathbb{R}^{n+1}$), then M is a hyperplane. If the image under the Gauss map is contained in a closed hemisphere, then M is a hyperplane or a cylinder over a self-shrinker of one dimension lower, see Theorem 3.1. The convex geometry of the sphere has been studied extensively by Jost–Xin–Yang [12]. Using their technique we could improve the first part of Theorem 3.1 and obtain Theorem 3.2, which is the best possible. The omitting range of the Gauss image would be the codimension one closed hemisphere $\overline{S}^{n-1}_+$, much smaller than the closed hemisphere $\overline{S}^n_+$ in Theorem 3.1.

In §4 we study the higher codimensional graphic situation. The target manifold of the Gauss map is the Grassmannian manifold now. To study the higher codimensional Bernstein problem, Jost–Xin–Yang [13] obtained some interesting geometric properties of the Grassmannian manifolds and developed some skilled technique. This enables us to obtain rigidity results of higher codimension for self-shrinkers. Using Theorem 3.1 in [13], Ding–Wang [4] obtained a result for this problem. Now, using the method of Theorem 3.1 in [13] we prove Proposition 4.1 to fit the present situation. Therefore, we obtain The-