Advances in Mathematics 303 (2016) 192–278

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Quasigeodesic pseudo-Anosov flows in hyperbolic 3-manifolds and connections with large scale geometry

Sérgio R. Fenley [a,b,*,1]

[a] Florida State University, Tallahassee, FL 32306-4510, United States
[b] Princeton University, Princeton, NJ 08544-1000, United States

A R T I C L E I N F O

Article history:
Received 18 May 2016
Accepted 20 May 2016
Available online 25 August 2016
Communicated by the Managing Editors of AIM

MSC:
primary 57M50, 37D20, 37D50, 37C85
secondary 20F67, 34D45, 57M60, 57R30

Keywords:
Hyperbolic dynamics
Pseudo-Anosov flows
Hyperbolic dynamics
Quasigeodesic flows

A B S T R A C T

In this article we obtain a simple topological and dynamical systems condition which is necessary and sufficient for an arbitrary pseudo-Anosov flow in a closed, hyperbolic three manifold to be quasigeodesic. Quasigeodesic means that orbits are efficient in measuring length up to a bounded multiplicative distortion when lifted to the universal cover. We prove that such flows are quasigeodesic if and only if there is an upper bound, depending only on the flow, on the number of orbits which are freely homotopic to an arbitrary closed orbit of the flow. The main ingredient is a proof that, under the boundedness condition, the fundamental group of the manifold acts as a uniform convergence group on a flow ideal boundary of the universal cover. We also construct a flow ideal compactification of the universal cover, and prove that it is equivariantly homeomorphic to the Gromov compactification. This implies the quasigeodesic behavior of the flow. The flow ideal boundary and flow ideal compactification are constructed using only the structure of the flow.

* Correspondence to: Florida State University, Tallahassee, FL 32306-4510, United States.
E-mail addresses: fenley@math.princeton.edu, fenley@math.fsu.edu.

[1] Research partially supported by a grant 280429 of the Simons Foundation.

http://dx.doi.org/10.1016/j.aim.2016.05.015

1. Introduction

The goal of this article is to relate dynamical systems behavior with the geometry of the underlying manifold, and in particular with the large scale geometry of the universal cover. We analyze pseudo-Anosov flows in three manifolds. This is an extremely common class of flows which is known to be closely related with the topology of the underlying manifold. We study these flows in hyperbolic three manifolds and we obtain a very simple characterization of good geometric behavior of the flow in the universal cover. We also prove that in the case of good geometric behavior the flow generates a flow ideal compactification of the universal cover which is equivariantly homeomorphic to the Gromov compactification. It follows that in these cases the flow encodes the asymptotic or large scale geometric structure of the universal cover.

The field of hyperbolic flows was started by Anosov [1] who studied geodesic flows in the unit tangent bundle of manifolds with negative sectional curvature. In fact Anosov studied much more general flows, which have since then been called Anosov flows. Anosov obtained deep and far reaching results concerning the dynamical behavior of these flows and with connections and applications to ergodic theory, foliation theory and other areas [1]. In dimension three these flows were generalized by Thurston who defined pseudo-Anosov homeomorphisms of surfaces [61] and used their suspensions to obtain deep results about three manifolds that fiber over the circle [59,60,62]. These suspension flows are the most basic examples of pseudo-Anosov flows.

Pseudo-Anosov flows are the most useful flows to study the topology of three manifolds [2,3,10–12,21,22,35,36,48–50]. The goal of this article is to establish a strong relationship between these flows and the geometry of the manifold, and more specifically with the large scale geometry of the universal cover. This is extremely important in the case of hyperbolic manifolds [37,39,59,60]. At first it might seem that nothing can be said in general about large scale geometric properties of flows lines. This is because flows and flow lines are very flexible and floppy and apparently not very geodesic. We will give a necessary and sufficient topological and dynamical systems condition for all the flow lines in the universal cover to have good geometric behavior.

Zeghib [65] proved that a flow in a closed hyperbolic 3-manifold cannot be geodesic, that is, not all flow lines can be geodesics in the hyperbolic metric. The next best property is that flow lines are quasigeodesics, which we now define. A *quasi-isometric embedding* is a map between metric spaces which is bi-Lipschitz in the large. Equivalently, up to an additive constant, the map is at most a bounded multiplicative distortion in the metric. A *quasigeodesic* is a quasi-isometric embedding of the real line (or a segment) into a metric space. The work of Thurston [59,60,62], Gromov [39] and many, many others have thoroughly established the fundamental importance of quasigeodesics in hyperbolic manifolds. In this article we analyze the interaction of the quasigeodesic property with flows in 3-manifolds.

Given a flow with rectifiable orbits in a manifold we say it is quasigeodesic if every flow line of the lifted flow to the universal cover is a quasigeodesic. The metric in the domain of