Advances in Mathematics 303 (2016) 611–637

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Twists versus modifications

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ARTICLE INFO

*Article history:*
Received 4 January 2016
Received in revised form 17 July 2016
Accepted 22 August 2016
Available online 30 August 2016
Communicated by Tony Pantev

*MSC:*
primary 53C26
secondary 31B05, 53D20, 57S25

*Keywords:*
HyperKähler
T-duality
Circle action
Moment map
Ricci-flat metric
Complete metric
Harmonic function
Martin boundary
Singularity
Strong HKT

ABSTRACT

The twist construction is a geometric T-duality that produces new manifolds from old, works well with for example hypercomplex structures and is easily inverted. It tends to destroy properties such as the hyperKähler condition. On the other hand modifications preserve the hyperKähler property, but do not have an obvious inversion. In this paper we show how elementary deformations provide a link between the two constructions, and use the twist construction to build hyperKähler and strong HKT structures. In the process, we provide a full classification of complete hyperKähler four-manifolds with tri-Hamiltonian symmetry and study a number singular phenomena in detail.



**Contents**



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http://dx.doi.org/10.1016/j.aim.2016.08.028
0001-8708/

## 1. Introduction

HyperKähler metrics are Ricci-flat with a triple of parallel complex structures. The metric together with any one of the complex structures specifies a Kähler geometry with parallel complex-symplectic form. Such manifolds are Calabi–Yau and form a special class in the Berger holonomy classification, see Besse [7].

Given an isometric circle action preserving each element in the triple, there are at least two different constructions that may be applied to produce manifolds in the same dimension with a new topology and a metric compatible with a triple of complex structures.

The first is the twist construction of [34], which reproduces the T-duality as used in Gibbons, Papadopoulos and Stelle [14]. In particular, it includes constructions of strong HKT metrics in dimension four from hyperKähler metrics. However, many of the examples discussed in [14] are incomplete, and it is not clear whether one can derive hyperKähler metrics from the construction. Indeed, while the twist construction in [34] is easily specialised to generate integrable complex structures, it does tend to destroy symplectic structures that are present. On the other hand the twist construction has the advantage that it is a genuine duality and may easily be inverted.

In contrast, the hyperKähler modification construction [11] produces hyperKähler manifolds in the same dimension via a hyperKähler moment map construction. When the original manifold is simply-connected, the modification increases the section Betti number by one. Away from the zero set of the moment map, the topological set-up is precisely a double fibration picture that is the basis for the twist construction. However, the recipe for producing the hyperKähler metric from this picture is rather different, and inversion is not apparent.

The purpose of this paper is to determine exactly how these constructions are related to each other, particularly when the metrics involved are complete. As the domain of the hyperKähler modification is larger than that of the double fibration, this also enables us to explore some singular behaviour of the twist construction.

We start the paper by giving a brief overview of the twist and modification constructions, describing their properties with respect to completeness. We also introduce a generalisation of the modification, which has as an ingredient an arbitrary complete hyperKähler four-manifold with circle symmetry in place of flat $\mathbb{H} = \mathbb{R}^4$ in the original