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On operator-valued bi-free distributions

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ABSTRACT

In this paper, operator-valued bi-free distributions are investigated. Given a subalgebra D of a unital algebra B, it is established that a two-faced family Z is bi-free from (B, B^{op}) over D if and only if certain conditions relating the B-valued and D-valued bi-free cumulants of Z are satisfied. Using this, we verify that a two-faced family of matrices is R-cyclic if and only if they are bi-free from the scalar matrices over the scalar diagonal matrices. Furthermore, the operator-valued bi-free partial R-, S-, and T-transforms are constructed. New proofs of results from free probability are developed in order to facilitate many of these bi-free results.

1. Introduction

The notion of bi-free pairs of faces was introduced by Voiculescu in [18] as a theory to enable the simultaneous study of the left and right actions of algebras on free product of vector spaces. Initially postulated by Mastnak and Nica in [7] and demonstrated by Charlesworth, Nelson, and the author in [3], the combinatorial structures for bi-free probability are bi-non-crossing partitions; partitions that are non-crossing once a specific

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permutation is applied. Consequently, as the combinatorics of free and bi-free probability are intimately related, many results from free probability have immediate generalizations to the bi-free setting. Other bi-free results, such as bi-free partial transforms (see [13,14, 19,20]) and bi-matrix models (see [15]), require additional work.

Although briefly examined in [18], operator-valued bi-free probability (a generalization of bi-free probability where the scalars $\mathbb{C}$ are replaced with an arbitrary unital algebra B) has received less attention. In [2], Charlesworth, Nelson, and the author demonstrated that the combinatorics of operator-valued bi-free probability is similar to the combinatorics of operator-valued free probability, yet additional care had to be taken.

Operator-valued free probability has been incredibly useful as its study enlarges the domain of mathematics where free probability techniques may be applied. This greater framework yields its own interesting results and allows additional techniques, which simplify arguments in free probability. Of course, the trade-off of this wider framework is that results are more difficult to obtain. These same ideas should resonate in operator-valued bi-free probability; the techniques should yield interesting results at the expense of the arguments being greater in difficulty. Furthermore, as bi-free probability is in its infancy, a further understanding of bi-free probability can be obtained through studying operator-valued bi-freeness since intuition from the operator-valued setting can yield results in the scalar setting (as was the case with the bi-matrix models in [15]).

Unfortunately, few concrete examples of bi-free pairs of B-faces are in existence to derive intuition from. The most natural example comes from considering a type II_1 factor $\mathfrak{M}$, a von Neumann subalgebra $\mathfrak{N}$ of $\mathfrak{M}$, and the conditional expectation $E_{\mathfrak{N}} : \mathfrak{M} \to \mathfrak{N}$ of $\mathfrak{M}$ onto $\mathfrak{N}$. To construct pairs of $\mathfrak{N}$-faces, consider the linear maps on $\mathfrak{M}$, denoted $\mathcal{L}(\mathfrak{M})$, and the expectation $E : \mathcal{L}(\mathfrak{M}) \to \mathfrak{N}$ defined by

$$E(T) = E_{\mathfrak{N}}(T(1_{\mathfrak{M}}))$$

for all $T \in \mathcal{L}(\mathfrak{M})$; that is, apply T to $1_{\mathfrak{M}}$ and take the expectation of $\mathfrak{M}$ onto $\mathfrak{N}$. Further define *-homomorphisms $L : \mathfrak{M} \to \mathcal{L}(\mathfrak{M})$ and $R : \mathfrak{M}^{\mathrm{op}} \to \mathcal{L}(\mathfrak{M})$ by

$$L(X)(A) = XA \qquad \text{and} \qquad R(X)(A) = AX$$

for all $X, A \in \mathfrak{M}$. For this discussion we will call $L(X)$ a left operator and $R(X)$ a right operator. If $\mathfrak{M} = \mathfrak{M}_1 *_{\mathfrak{N}} \mathfrak{M}_2$, the amalgamated free product of von Neumann algebras $\mathfrak{M}_1$ and $\mathfrak{M}_2$ over $\mathfrak{N}$, then the pairs of $\mathfrak{N}$-faces $(L(\mathfrak{M}_1), R(\mathfrak{M}_1^{\mathrm{op}}))$ and $(L(\mathfrak{M}_2), R(\mathfrak{M}_2^{\mathrm{op}}))$ are bi-free with amalgamation over $\mathfrak{N}$ with respect to E.

Like free probability, bi-free probability is a theory which describes the joint moments of operators. In particular, free probability may be viewed as a subcase of bi-free probability where only left operators are considered thereby providing intuition for the bi-free case. Recall, to compute the $\mathfrak{N}$-distribution of elements of $\mathfrak{M}$, given $X_1, \ldots, X_n \in \mathfrak{M}$ one takes $T_1, \ldots, T_n, T_1', \ldots, T_n' \in \mathfrak{N}$ and computes the joint moment of

$$L(T_1)L(X_1)L(T_1'), \ldots, L(T_n)L(X_n)L(T_n'),$$