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Noncommutative motives of separable algebras

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ABSTRACT

In this article we study in detail the category of noncommutative motives of separable algebras Sep(k) over a base field k. We start by constructing four different models of the full subcategory of commutative separable algebras CSep(k). Making use of these models, we then explain how the category Sep(k) can be described as a “fibered $\mathbb{Z}$-order” over CSep(k). This viewpoint leads to several computations and structural properties of the category Sep(k). For example, we obtain a complete dictionary between directs sums of noncommutative motives of central simple algebras (= CSA) and sequences of elements in the Brauer group of k. As a first application, we establish two families of motivic relations between CSA which hold for every additive invariant (*e.g.* algebraic K-theory, cyclic homology, and topological Hochschild homology). As a second application, we compute the additive invariants of twisted flag varieties using solely the Brauer classes of the corresponding CSA. Along the way, we categorify the cyclic sieving phenomenon and compute the (rational) noncommu-

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Cyclic sieving phenomenon
dg Azumaya algebra

tative motives of purely inseparable field extensions and of dg Azumaya algebras.

1. Introduction

Noncommutative motives

A *dg category* $\mathcal{A}$, over a base field k, is a category enriched over complexes of k-vector spaces; see §4. Every (dg) k-algebra A naturally gives rise to a dg category with a single object. Another source of examples is provided by schemes since the category of perfect complexes $\mathrm{perf}(X)$ of every quasi-compact quasi-separated k-scheme X admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$. In what follows, $\mathrm{dgcat}(k)$ denotes the category of dg categories.

Invariants such as algebraic K-theory, cyclic homology, and topological Hochschild homology, extend naturally from k-algebras to dg categories. In order to study them simultaneously the notion of *additive invariant* was introduced in [31] and the *universal additive invariant* $U : \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$ was constructed; consult §5.1–5.2 for details. Given any additive category D, there is an induced equivalence

$$U^* : \mathrm{Fun}(\mathrm{Hmo}_0(k), \mathrm{D}) \xrightarrow{\simeq} \mathrm{Fun}_{\mathrm{add}}(\mathrm{dgcat}(k), \mathrm{D})\,, \tag{1.1}$$

where the left-hand-side denotes the category of additive functors and the right-hand-side the category of additive invariants. Because of this universal property, which is reminiscent from the yoga of motives, $\mathrm{Hmo}_0(k)$ is called the category of *noncommutative motives*. The tensor product of k-algebras extends also naturally to dg categories. It gives rise to a symmetric monoidal structure on $\mathrm{dgcat}(k)$ which descends to $\mathrm{Hmo}_0(k)$ making the functor U symmetric monoidal.

Following Kontsevich [17–19], a dg category $\mathcal{A}$ is called *smooth* if it is perfect as a bimodule over itself and *proper* if $\sum_i \dim H^i\mathcal{A}(x,y) < \infty$ for every ordered pair of objects (x,y). Examples include the finite dimensional k-algebras of finite global dimension (when k is perfect) and the dg categories $\mathrm{perf}_{\mathrm{dg}}(X)$ associated to smooth projective k-schemes X. The category of *noncommutative Chow motives* $\mathrm{NChow}(k)$ was introduced in [33] as the idempotent completion of the full subcategory of $\mathrm{Hmo}_0(k)$ consisting of the smooth proper dg categories. By construction, $\mathrm{NChow}(k)$ is additive and rigid symmetric monoidal; consult the survey [32, §4].

Motivating goal

Given an additive rigid symmetric monoidal category $\mathcal{C}$, its $\otimes$-ideal $\mathcal{N}$ is defined by the following formula

$$\mathcal{N}(a,b) := \{ f : a \to b \,|\, \forall\ g : b \to a \ \text{ we have } \ \mathrm{tr}(g \circ f) = 0 \}\,,$$