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Pure dimension and projectivity of tropical polytopes

Zur Izhakian [a,1], Marianne Johnson [b,2], Mark Kambites [b,*,3]

[a] *Institute of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, UK*
[b] *School of Mathematics, University of Manchester, Manchester M13 9PL, UK*

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ABSTRACT

We study how geometric properties of tropical convex sets and polytopes, which are of interest in many application areas, manifest themselves in their algebraic structure as modules over the tropical semiring. Our main results establish a close connection between *pure dimension* of tropical convex sets, and *projectivity* (in the sense of ring theory). These results lead to a geometric understanding of idempotency for tropical matrices. As well as their direct interest, our results suggest that there is substantial scope to apply ideas and techniques from abstract algebra (in particular, ring theory) in tropical geometry.

* Corresponding author.
E-mail addresses: zzur@abdn.ac.uk (Z. Izhakian), Marianne.Johnson@maths.manchester.ac.uk (M. Johnson), Mark.Kambites@manchester.ac.uk (M. Kambites).

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1. Introduction

Tropical mathematics can be loosely defined as the study of the real numbers (sometimes augmented with $-\infty$) under the operations of addition and maximum (or equivalently, minimum). It has been an active area of study in its own right since the 1970's [13] and also has well-documented applications in diverse areas such as analysis of discrete event systems, control theory, combinatorial optimisation and scheduling problems [27], formal languages and automata [29], phylogenetics [16], statistical inference [28], combinatorial/geometric group theory [5] and most recently in algebraic geometry (see for example [21]). A key role in most of these areas is played by *tropically convex sets* and *tropical polytopes*. These subsets of tropical space are naturally endowed not only with a geometric structure (as Euclidean polyhedral complexes), but also with a purely algebraic structure (as modules over the tropical semiring).

In this paper we consider the way in which geometric properties of convex sets and polytopes, of interest in many application areas, manifest themselves in their algebraic structure. Our main results establish a close connection between *pure dimension* (a geometric property of interest for applications of tropical methods) and *projectivity* (in the sense of ring theory and category theory). These results lead to a geometric understanding of idempotency for tropical matrices. As a corollary, we obtain the fact that all the widely studied notions of *rank* for tropical matrices coincide where the matrices are idempotent or, more generally, von Neumann regular. (This fact was mentioned without proof in [1, Fact 4, Section 35.7], with reference given to a preprint of Cohen, Gaubert and Quadrat which at the time of writing is still not available to the public.) As well as their direct interest, these results suggest that there is substantial scope to apply ideas and techniques from abstract algebra (in particular, ring theory) to understand problems in tropical geometry.

We denote by $\mathbb{FT}$ the *(finitary) tropical semiring*, which consists of the real numbers under the operations of addition and maximum. We write $a \oplus b$ to denote the maximum of a and b, and $a \otimes b$ or just ab to denote the sum of a and b. It is readily checked that both operations are associative and commutative, $\otimes$ has a neutral element (0), admits inverses and distributes over $\oplus$, while $\oplus$ is *idempotent* ($a \oplus a = a$ for all a). These properties mean that $\mathbb{FT}$ has the structure of an *idempotent semifield (without zero)*.

The space $\mathbb{FT}^n$ of tropical n-vectors admits natural operations of componentwise maximum and the obvious scaling by $\mathbb{FT}$, which makes it into an $\mathbb{FT}$-*module*. It also has a natural partial order. For detailed definitions see Section 2 below. Submodules of $\mathbb{FT}^n$ play a vital role in tropical mathematics; as well as their obvious algebraic importance, they have a geometric structure in view of which they are usually called *(tropical) convex sets* or sometimes *convex cones*. Particularly important are the finitely generated convex sets, which are called *(tropical) polytopes*.

There are several important notions of *dimension* for convex sets. The *(affine) tropical dimension* is the topological dimension of the set, viewed as a subset of $\mathbb{R}^n$ with the usual topology. The *projective tropical dimension* (sometimes just called *dimension* in