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Autoequivalences of derived categories via geometric invariant theory

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ABSTRACT

We study autoequivalences of the derived category of coherent sheaves of a variety arising from a variation of GIT quotient. We show that these autoequivalences are spherical twists, and describe how they result from mutations of semiorthogonal decompositions. Beyond the GIT setting, we show that all spherical twist autoequivalences of a dg-category can be obtained from mutation in this manner.

Motivated by a prediction from mirror symmetry, we refine the recent notion of "grade restriction rules" in equivariant derived categories. We produce additional derived autoequivalences of a GIT quotient and propose an interpretation in terms of monodromy of the quantum connection. We generalize this observation by proving a criterion under which a spherical twist autoequivalence factors into a composition of other spherical twists.

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1. Introduction

Homological mirror symmetry predicts, in certain cases, that the bounded derived category of coherent sheaves on an algebraic variety should admit *twist autoequivalences* corresponding to a spherical object [17]. The autoequivalences predicted by mirror symmetry have been widely studied, and the notion of a spherical object has been generalized to the notion of a spherical functor [1] (see Definition 3.10). We apply recently developed techniques for studying the derived category of a geometric invariant theory (GIT) quotient [5,10,12,13,16] to the construction of autoequivalences, and our investigation leads to general connections between the theory of spherical functors and the theory of semiorthogonal decompositions and mutations.

We consider an algebraic stack which arises as a GIT quotient of a smooth quasiprojective variety X by a reductive group G. By varying the G-ample line bundle used to define the semistable locus, one gets a birational transformation $X^{ss}_-/G \dashrightarrow X^{ss}_+/G$ called a variation of GIT quotient (VGIT). We study a simple type of VGIT, which we call a *balanced wall crossing* (see Section 3).

Under a hypothesis on ω_X, a balanced wall crossing gives rise to an equivalence $\psi_w : \mathrm{D}^b(X^{ss}_-/G) \to \mathrm{D}^b(X^{ss}_+/G)$ which depends on a choice of $w \in \mathbb{Z}$, and the composition $\Phi_w := \psi_{w+1}^{-1}\psi_w$ defines an autoequivalence of $\mathrm{D}^b(X^{ss}_-/G)$. Autoequivalences of this kind have been studied recently under the name window-shifts [10,16]. We generalize the observations of those papers in showing that Φ_w is always a spherical twist.

Recall that if B is an object in a dg-category, then we can define the twist functor

$$T_B : F \mapsto \mathrm{Cone}(\mathrm{Hom}^\bullet(B, F) \otimes_{\mathbb{C}} B \to F)$$

If B is a spherical object, then T_B is by definition the spherical twist autoequivalence defined by B. More generally, if $S : \mathcal{A} \to \mathcal{B}$ is a spherical functor (Definition 3.10), then one can define a twist autoequivalence $T_S := \mathrm{Cone}(S \circ S^R \to \mathrm{id}_{\mathcal{B}})$ of $\mathcal{B}$, where S^R denotes the right adjoint. Throughout this paper we refer to a twist autoequivalence