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Uniqueness and properties of distributional solutions of nonlocal equations of porous medium type

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ABSTRACT

We study the uniqueness, existence, and properties of bounded distributional solutions of the initial value problem for the anomalous diffusion equation $\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0$. Here $\mathcal{L}^\mu$ can be any nonlocal symmetric degenerate elliptic operator including the fractional Laplacian and numerical discretizations of this operator. The function $\varphi : \mathbb{R} \to \mathbb{R}$ is only assumed to be continuous and nondecreasing. The class of equations include nonlocal (generalized) porous medium equations, fast diffusion equations, and Stefan problems. In addition to very general uniqueness and existence results, we obtain stability, L^1-contraction, and a priori estimates. We also study local limits, continuous dependence, and properties and convergence of a numerical approximation of our equations.

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Numerical approximation
Convergence

1. Introduction

In this paper, we obtain uniqueness, existence, and various other properties for bounded distributional solutions of a class of possibly degenerate nonlinear anomalous diffusion equations of the form:

$$\partial_t u - \mathcal{L}^\mu[\varphi(u)] = 0 \qquad \text{in} \quad Q_T := \mathbb{R}^N \times (0,T), \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad \text{on} \quad \mathbb{R}^N, \tag{1.2}$$

where $u = u(x,t)$ is the solution and $T > 0$. The nonlinearity φ is an arbitrary continuous nondecreasing function, while the anomalous or nonlocal diffusion operator $\mathcal{L}^\mu$ is defined for any $\psi \in C_c^\infty(\mathbb{R}^N)$ as

$$\mathcal{L}^\mu[\psi](x) = \int_{\mathbb{R}^N\setminus\{0\}} \Big(\psi(x+z) - \psi(x) - z\cdot D\psi(x)\mathbf{1}_{|z|\leq 1}\Big)\,\mathrm{d}\mu(z), \tag{1.3}$$

where D is the gradient, $\mathbf{1}_{|z|\leq 1}$ a characteristic function, and μ a nonnegative symmetric possibly singular measure satisfying the Lévy condition $\int |z|^2 \wedge 1\,\mathrm{d}\mu(z) < \infty$. For the precise assumptions, we refer to Section 2.

The class of nonlocal diffusion operators we consider coincide with the generators of the symmetric pure-jump Lévy processes [8,6,38] like e.g. compound Poisson processes, CGMY processes in Finance, and symmetric s-stable processes. Included are the well-known fractional Laplacians $-(-\Delta)^{\frac{s}{2}}$ for $s \in (0,2)$ (where $\mathrm{d}\mu(z) = c_{N,s}\frac{\mathrm{d}z}{|z|^{N+s}}$ for some $c_{N,s} > 0$ [23,6]), along with degenerate operators, and surprisingly, numerical discretizations of these operators!

In the language of [47], equation (1.1) is a generalized porous medium equation. On one hand, since φ is only assumed to be continuous, the full range of porous medium and fast diffusion nonlinearities are included: $\varphi(r) = r|r|^{m-1}$ for $m > 0$. This is somehow optimal for power nonlinearities since if $m < 0$ (ultra fast diffusion), then not only uniqueness, but also existence may fail [11]. On the other hand, since φ is only assumed to be nondecreasing, it can be constant on sets of positive measure and then equation (1.1) is strongly degenerate. This case include Stefan type of problems, like e.g. when $c_1, c_2, T > 0$ and