Advances in Mathematics 305 (2017) 339–350

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

On a question of R.H. Bing concerning the fixed point property for two-dimensional polyhedra

Jonathan Ariel Barmak [*,1], Iván Sadofschi Costa

Departamento de Matematica, FCEyN-Universidad de Buenos Aires, Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 1 April 2015
Received in revised form 12 September 2016
Accepted 13 September 2016
Available online xxxx
Communicated by Tomasz S. Mrowka

MSC:
55M20
57M20
55P15
57M05

Keywords:
Fixed point property
Two-dimensional complexes
Nielsen fixed point theory
Homotopy classification

ABSTRACT

In 1969 R.H. Bing asked the following question: Is there a compact two-dimensional polyhedron with the fixed point property which has even Euler characteristic? In this paper we prove that there are no spaces with these properties and abelian fundamental group. We also show that the fundamental group of such a complex cannot have trivial Schur multiplier.

* Corresponding author.
E-mail addresses: jbarmak@dm.uba.ar (J.A. Barmak), isadofschi@dm.uba.ar (I. Sadofschi Costa).

[1] Researcher of CONICET. Partially supported by grants ANPCyT PICT-2011-0812, CONICET PIP 112-201101-00746 and UBA UBACyT 20020130100369.

http://dx.doi.org/10.1016/j.aim.2016.09.025

1. Introduction

In his famous article "The elusive fixed point property" of 1969 [1], R.H. Bing stated twelve questions which were a motivation for the development of several methods in the fixed point theory of polyhedra and continua. In the last 45 years, eight of these questions have been answered while four of them remain still open [7]. In this paper we make an advance on Bing's Question 1.

Recall that a space X is said to have the *fixed point property* if every continuous self-map $f : X \to X$ has a fixed point. The fixed point property is clearly a topological invariant, but W. Lopez showed in [12] that it is not a homotopy invariant in the category of compact polyhedra (although it is *almost* a homotopy invariant, see Theorem 2.2 below). In order to provide an example Lopez constructed an eight-dimensional polyhedron with the fixed point property and even Euler characteristic. This example motivated the following question.

Question 1.1 *(Bing's Question 1). Is there a compact two-dimensional polyhedron with the fixed point property which has even Euler characteristic?*

It is not hard to find examples of two-dimensional complexes with the fixed point property. For any finite group G with deficiency equal to 0, there exists a 2-complex X with $\pi_1(X) = G$ and $\widetilde{H}_*(X;\mathbb{Q}) = 0$. By the Lefschetz fixed point theorem, any compact two-dimensional polyhedron X with trivial rational homology has the fixed point property. However, it is unknown whether the converse of the latter statement holds. Therefore we will also consider the following variation of Bing's question:

Question 1.2. *Is there a compact two-dimensional polyhedron with the fixed point property such that $\widetilde{H}_*(X;\mathbb{Q}) \neq 0$?*

Of course, an affirmative answer to Question 1.1 implies an affirmative answer to Question 1.2.

It is well-known that a polyhedron X with $H_1(X;\mathbb{Q}) \neq 0$ lacks the fixed point property since S^1 is a retract of any such space. Therefore, for a compact 2-complex with the fixed point property it is equivalent to saying that $\widetilde{H}_*(X;\mathbb{Q}) \neq 0$, that $\chi(X) > 1$ or that the integer homology group $H_2(X) \neq 0$.

A higher dimensional analogue to Question 1.2 has been settled by Waggoner [14] for dimension $n \geq 4$ and later extended to dimension 3 by Jiang [10].

Theorem 1.3 *(Waggoner, Jiang). If X is a compact $(n-2)$-connected polyhedron of dimension $n > 2$ and $\widetilde{H}_*(X;\mathbb{Q}) \neq 0$, then X does not have the fixed point property.*

In this article a compact two-dimensional polyhedron will be called a *Bing space* if it has the fixed point property and $\widetilde{H}_*(X;\mathbb{Q}) \neq 0$. The main result of this paper is the following