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# Potentials and Chern forms for Weil–Petersson and Takhtajan–Zograf metrics on moduli spaces

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A B S T R A C T

For the TZ metric on the moduli space $\mathscr{M}_{0,n}$ of $n$-pointed rational curves, we construct a Kähler potential in terms of the Fourier coefficients of the Klein's Hauptmodul. We define the space $\mathfrak{S}_{g,n}$ as holomorphic fibration $\mathfrak{S}_{g,n} \to \mathfrak{S}_g$ over the Schottky space $\mathfrak{S}_g$ of compact Riemann surfaces of genus $g$, where the fibers are configuration spaces of $n$ points. For the tautological line bundles $\mathscr{L}_i$ over $\mathfrak{S}_{g,n}$, we define Hermitian metrics $h_i$ in terms of Fourier coefficients of a covering map $J$ of the Schottky domain. We define the regularized classical Liouville action $S$ and show that $\exp\{S/\pi\}$ is a Hermitian metric in the line bundle $\mathscr{L} = \otimes_{i=1}^{n} \mathscr{L}_i$ over $\mathfrak{S}_{g,n}$. We explicitly compute the Chern forms of these Hermitian line bundles

$$c_1(\mathscr{L}_i, h_i) = \frac{4}{3}\omega_{\mathrm{TZ},i}, \quad c_1(\mathscr{L}, \exp\{S/\pi\}) = \frac{1}{\pi^2}\omega_{\mathrm{WP}}.$$

We prove that a smooth real-valued function $-\mathscr{S} = -S + \pi\sum_{i=1}^{n}\log h_i$ on $\mathfrak{S}_{g,n}$, a potential for this special difference of WP and TZ metrics, coincides with the renormalized hyper-

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bolic volume of a corresponding Schottky 3-manifold. We extend these results to the quasi-Fuchsian groups of type $(g, n)$.


**Contents**



## 1. Introduction

Weil introduced the Weil–Petersson (WP) metric on the moduli spaces of Riemann surfaces by using the Petersson inner product on the holomorphic cotangent spaces, the complex vector spaces of cusp forms of weight 4. Ahlfors proved that the WP metric is Kähler and its Ricci, holomorphic sectional and scalar curvatures are all negative [1,2], and Wolpert found a closed formula for the Riemann tensor of the WP metric and obtained explicit bounds for its curvatures [17].

In [19,20] it was shown that for the moduli space $\mathscr{M}_{0,n}$ of marked Riemann surfaces of type $(0, n)$, $n > 3$ ($n$-pointed rational curves) and for the Schottky space $\mathfrak{S}_g$ of compact Riemann surfaces of genus $g > 1$ the WP metric has global Kähler potential, the so-called *classical Liouville action* (for precise definitions, see Sects. 2 and 3). In [12,13] a new Kähler metric was introduced on the moduli space $\mathfrak{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n > 0$ punctures, $3g - 3 + n > 0$. In [8,10,11,16,18] it was called Takhtajan–Zograf (TZ) metric (for its precise definition, see Sect. 2.1.2). Unlike the WP metric, the curvature properties of the TZ metric are not known.

Here we present explicit formula for a Kähler potential $h_i$ of the $i$-th TZ metric on the moduli space $\mathscr{M}_{0,n}$, $i = 1, \ldots, n$. Specifically, in Proposition 1 we prove that $h_i$ is