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A Hopf algebra of subword complexes

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We introduce a Hopf algebra structure of subword complexes, including both finite and infinite types. We present an explicit cancellation free formula for the antipode using acyclic orientations of certain graphs, and show that this Hopf algebra induces a natural non-trivial sub-Hopf algebra on c-clusters in the theory of cluster algebras.

1. Introduction

Subword complexes are simplicial complexes introduced by Knutson and Miller, and are motivated by the study of Gröbner geometry of Schubert varieties [23,24]. These complexes have been shown to have striking connections with diverse topics such as associahedra [27,41,42], multi-associahedra [22,40], pseudotriangulation polytopes [37,38], and cluster algebras [16,17].

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The first connection between subword complexes and associahedra was discovered by Pilaud and Pocchiola who showed that every multi-associahedron can be obtained as a well chosen type A subword complex in the context of sorting networks [29]. A particular instance of their result was rediscovered using the subword complex terminology in [39, 44]. These results were generalized to arbitrary finite Coxeter groups by Ceballos, Labbé and Stump in [7]. The results in [7] provide an additional connection with the c-cluster complexes in the theory of cluster algebras, which has been used as a keystone for explicit results about denominator vectors in cluster algebras of finite type [9]. A construction of certain brick polytopes of spherical subword complexes is presented in [31,32], which is used to give a precise description of the toric varieties of c-generalized associahedra in connection with Bott–Samelson varieties in [14]. More recent developments on geometric and combinatorial properties of subword complexes are presented in [4,15,45].

This paper presents a more algebraic approach to subword complexes. We introduce a Hopf algebra structure on the vector space generated by all facets of irreducible subword complexes, including both finite and infinite types. Such facets include combinatorial objects such as triangulations and multi-triangulations of convex polygons, pseudotriangulations of any planar point set in general position, and c-clusters in cluster algebras of finite type. We present an explicit cancellation-free formula for the antipode using acyclic orientations of certain graphs. It is striking to observe that we obtain a result very similar to the antipode formula of Humpert and Martin for the incidence Hopf algebra of graphs [20]. As in [3], our combinatorial Hopf algebra is part of a nice family with explicit cancellation-free formula for the antipode. The Hopf algebra of subword complexes also induces a natural sub-Hopf algebra on c-clusters of finite type. Cluster complexes for Weyl groups were introduced by Fomin and Zelevinsky in connection with their proof of Zamolodchikov's periodicity conjecture for algebraic Y-systems in [18]. These complexes encode the combinatorial structure behind the associated cluster algebra of finite type [17], and are further extended to arbitrary Coxeter groups by Reading in [34]. The resulting c-cluster complexes use a Coxeter element c as a parameter and have been extensively used to produce geometric constructions of generalized associahedra [19,32,35,43]. The basis elements of our Hopf algebra of c-clusters are given by pairs of clusters (A, T) of finite type, where A is any acyclic cluster seed and T is any cluster obtained from it by mutations. The multiplication and comultiplication operations are natural from the cluster algebra perspective on T. However, subword complexes allow us to nontrivially extend these operations to remarkable operations on the acyclic seed A.

Our initial motivation was to extend the Loday–Ronco Hopf algebra on planar binary trees [25] in the context of subword complexes, and to present an algebraic approach to subword complexes that helps to better understand their geometry. The description of the Loday–Ronco Hopf algebra in terms of subword complexes of type A will be presented in a forthcoming paper in joint work with Pilaud [5]. The Hopf algebra described in this paper does not recover the Loday–Ronco Hopf algebra and differs from our original intent for several reasons: it allows an extension to arbitrary Coxeter groups, it restricts well to the context of c-clusters, and it contains more geometric information