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Explicit Serre duality on complex spaces

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A B S T R A C T

In this paper we use recently developed calculus of residue currents together with integral formulas to give a new explicit analytic realization, as well as a new analytic proof, of Serre duality on any reduced pure n-dimensional paracompact complex space X. At the core of the paper is the introduction of certain fine sheaves $\mathscr{B}_X^{n,q}$ of currents on X of bidegree (n,q), such that the Dolbeault complex $(\mathscr{B}_X^{n,\bullet}, \bar\partial)$ becomes, in a certain sense, a dualizing complex. In particular, if X is Cohen–Macaulay then $(\mathscr{B}_X^{n,\bullet}, \bar\partial)$ is an explicit fine resolution of the Grothendieck dualizing sheaf.

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1. Introduction

Let X be a complex n-dimensional manifold and let $F \to X$ be a complex vector bundle. Let $\mathcal{E}^{0,q}(X,F)$ denote the space of smooth F-valued $(0,q)$-forms on X and let $\mathcal{E}_c^{n,q}(X,F^*)$ denote the space of smooth compactly supported (n,q)-forms on X with values in the dual vector bundle F^*. Serre duality, [29], can be formulated analytically as follows: *There is a non-degenerate pairing*

$$H^q\left(\mathcal{E}^{0,\bullet}(X,F),\bar\partial\right) \times H^{n-q}\left(\mathcal{E}_c^{n,\bullet}(X,F^*),\bar\partial\right) \to \mathbb{C}, \qquad (1.1)$$
$$([\varphi]_{\bar\partial},[\psi]_{\bar\partial}) \mapsto \int_X \varphi\wedge\psi,$$

provided that $H^q(\mathcal{E}^{0,\bullet}(X,F),\bar\partial)$ and $H^{q+1}(\mathcal{E}^{0,\bullet}(X,F),\bar\partial)$ are Hausdorff considered as topological vector spaces. If we set $\mathscr{F} := \mathscr{O}(F)$ and $\mathscr{F}^* := \mathscr{O}(F^*)$ and let Ω_X^n denote the sheaf of holomorphic n-forms on X, then one can, via the Dolbeault isomorphism, rephrase Serre duality more algebraically: There is a non-degenerate pairing

$$H^q(X,\mathscr{F}) \times H_c^{n-q}(X,\mathscr{F}^*\otimes\Omega_X^n) \to \mathbb{C}, \qquad (1.2)$$

realized by the cup product, provided that $H^q(X,\mathscr{F})$ and $H^{q+1}(X,\mathscr{F})$ are Hausdorff. In this formulation Serre duality has been generalized to complex spaces, see, e.g., Hartshorne [20,19], and Conrad [16] for the algebraic setting and Ramis–Ruget [27] and Andreotti–Kas [11] for the analytic. In fact, if X is a pure n-dimensional paracompact complex space that in addition is Cohen–Macaulay, then again there is a perfect pairing (1.2) if we construe Ω_X^n as the *Grothendieck dualizing sheaf* that we will get back to shortly. If X is not Cohen–Macaulay things get more involved and $H_c^{n-q}(X,\mathscr{F}^*\otimes\Omega_X^n)$ is replaced by $\mathrm{Ext}_c^{-q}(X;\mathscr{F},\mathbf{K}^\bullet)$, where $\mathbf{K}^\bullet$ is the *dualizing complex* in the sense of Ramis–Ruget [27]; that is a certain complex of $\mathscr{O}_X$-modules with coherent cohomology.

To our knowledge there is no such explicit analytic realization of Serre duality as (1.1) in the case of singular spaces. In fact, *verbatim* the pairing (1.1) cannot realize Serre duality in general since the Dolbeault complex $(\mathcal{E}_X^{0,\bullet},\bar\partial)$[3] in general does not provide a resolution of $\mathscr{O}_X$. In this paper we replace the sheaves of smooth forms by certain fine sheaves of currents $\mathscr{A}_X^{0,q}$ and $\mathscr{B}_X^{n,n-q}$ that are smooth on X_{reg} and such that (1.1) with $\mathcal{E}^{0,\bullet}$ and $\mathcal{E}^{n,\bullet}$ replaced by $\mathscr{A}^{0,\bullet}$ and $\mathscr{B}^{n,\bullet}$, respectively, indeed realizes Serre duality.

We will say that a complex $(\mathscr{D}_X^\bullet,\delta)$ of fine sheaves is a *dualizing Dolbeault complex* for a coherent sheaf $\mathscr{F}$ if $(\mathscr{D}_X^\bullet,\delta)$ has coherent cohomology and if there is a non-degenerate pairing $H^q(X,\mathscr{F})\times H^{n-q}(\mathscr{D}_c^\bullet(X),\delta)\to\mathbb{C}$. The relation to the Ramis–Ruget dualizing complex is not completely clear to us, but we still find this terminology convenient. For instance, $(\mathscr{B}_X^{n,\bullet},\bar\partial)$ is a dualizing Dolbeault complex for $\mathscr{O}_X$.

[3] See below for the definition of $\mathcal{E}_X^{p,q}$; the sheaf of smooth (p,q)-forms on X.