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Enumeration of rational curves with cross-ratio constraints

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ABSTRACT

In this paper we prove the algebraic–tropical correspondence for stable maps of rational curves with marked points to toric varieties such that the marked points are mapped to given orbits in the big torus and in the boundary divisor, the map has prescribed tangency to the boundary divisor, and certain quadruples of marked points have prescribed cross-ratios. In particular, our results generalize the results of Nishinou–Siebert [14]. The proof is very short, involves only the standard theory of schemes, and works in arbitrary characteristic (including the mixed characteristic case).

Contents

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1. Introduction

1.1. Background

Enumeration of curves in algebraic varieties is a classical problem that has a long history going back to Ancient Greeks. Many tools have been developed to approach enumerative problems including Schubert calculus, intersection theory, degeneration techniques, quantum cohomology etc.

In 1989, Ran [16] proposed a recursive procedure based on degeneration techniques for enumeration of nodal curves of given degree and genus in the plane satisfying point constraints. Several years later, there was a major break-through in the problem, when Kontsevich introduced the moduli spaces of stable maps [10,9], and used them to get recursive formulae for the number of rational plane curves of degree d passing through $3d-1$ points in general position. Kontsevich interpreted the enumerative invariants as Gromov–Witten invariants, and deduced information about them from properties of the quantum cohomology.

Stated in more classical terms, Kontsevich's recursion can be obtained as follows: one considers the curves with $3d$ marked points such that the first two are mapped to two given lines, the remaining $3d-2$ to points in general position, and the cross-ratio with respect to the first four points is a parameter. Specialization of the cross ratio to the values 0 and 1 then gives the desired recursion.

Using various degeneration techniques recursive formulae for different target spaces and higher genera have been obtained in a series of works of Pandharipande [15], Caporaso–Harris [3,4], Vakil [22], and others.

In the beginning of the century, Mikhalkin [12] proposed a new approach to the problem. He introduced piece-wise linear objects, called tropical curves, and proved that there is a natural correspondence between algebraic and tropical curves, that in good cases allows one to obtain closed formulae for the number of algebraic curves in terms