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Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group ☆

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ABSTRACT

We prove Hardy inequalities for the conformally invariant fractional powers of the sublaplacian on the Heisenberg group $\mathbb{H}^n$. We prove two versions of such inequalities depending on whether the weights involved are non-homogeneous or homogeneous. In the first case, the constant arising in the Hardy inequality turns out to be optimal. In order to get our results, we will use ground state representations. The key ingredients to obtain the latter are some explicit integral representations for the fractional powers of the sublaplacian and a generalized result by M. Cowling and U. Haagerup. The approach to prove the integral representations is via the language of semigroups. As a consequence of the Hardy inequalities we also obtain versions of Heisenberg uncertainty inequality for the fractional sublaplacian.

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1. Introduction and main results

The study and understanding of various kinds of weighted and unweighted inequalities for differential operators and the Fourier transform have been a matter of intensive research. This interest has been triggered and sustained by the importance of such inequalities in applications to problems in analysis, mathematical physics, spectral theory, fluid mechanics and stability of matter. Moreover, the sharpness of the constants involved in these inequalities is the key in establishing existence and non-existence results for certain non-linear Schrödinger equations.

For instance, the Pitt's inequality, the Hardy–Littlewood–Sobolev inequality and the logarithmic Sobolev inequality are in connection with the measure of uncertainty [6,8,9]. The Sobolev, Hardy, or Hardy–Sobolev type inequalities are applied to prove stability of relativistic matter (see [21]). They also deliver insight on the geometric structure of the space considered, and the knowledge of the best constants also helps to solve isoperimetric inequalities or decide the existence of solutions of certain PDE's, see [11] for a description of these topics.

A lot of work concerning these inequalities has been developed in the context of the Euclidean space and Riemannian manifolds, but not very much has been done in the framework of subriemannian geometry, in particular in the Heisenberg group. We refer to the remarkable work by R.L. Frank and E.H. Lieb [20] where they derive sharp constants for the Hardy–Littlewood–Sobolev inequalities on the Heisenberg group. We also refer the reader to [3,7,15,22] concerning several kinds of inequalities related to either the Grushin operator, or in Carnot–Carathéodory spaces, or on the Heisenberg group. There is a vast literature on this topic and our bibliography refers only to a very small fraction of the articles dealing with such inequalities and their applications.

In this article we are concerned with Hardy-type inequalities for the conformally invariant (or covariant, both nomenclatures seem to be used with the same meaning in the literature) fractional powers of the sublaplacian $\mathcal{L}$ on the Heisenberg group $\mathbb{H}^n$. Some Hardy inequalities are already known for the sublaplacian, see for instance [2,5,22], and also the very recent work by P. Ciatti, M. Cowling and F. Ricci [13] (see Remark 1.7 below). However, in [5] and [13] where the fractional powers are treated, the authors have not paid attention to the sharpness of the constants.

To begin with, let us recall two inequalities in the case of the Laplacian $\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ on $\mathbb{R}^n$. First, the standard Sobolev embedding $W^{s/2,2}(\mathbb{R}^n) \hookrightarrow L^{2n/(n-s)}(\mathbb{R}^n)$ for $0 < s < n$ leads to the optimal inequality

$$\|f\|_q^2 \leq c_{n,s}\langle \Delta^{s/2} f, f\rangle \tag{1.1}$$

with $c_{n,s} = \omega_n^{-s/n}\frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{n+s}{2})}$ where ω_n is the surface measure of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ and $q = (2n)/(n-s)$. Here and later, the symbol $\langle\cdot,\cdot\rangle$ denotes the inner product in the corresponding space. The above inequality is usually referred to as the Hardy–Littlewood–Sobolev (HLS) inequality for the fractional Laplacian $\Delta^{s/2}$ in the literature.