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Canonical bases for Fock spaces and tensor products

Joseph Chuang [a], Kai Meng Tan [b,*,1]

[a] *Department of Mathematics, City University London, Northampton Square, London EC1V 0HB, United Kingdom*
[b] *Department of Mathematics, National University of Singapore, Block S17, 10 Lower Kent Ridge Road, Singapore 119076, Singapore*

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ABSTRACT

We relate the canonical basis of the Fock space representation of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$, as defined by Leclerc and Thibon [15], to the canonical basis of its restriction to $U_q(\mathfrak{sl}_n)$, regarded as a based module in the sense of Lusztig. More generally we consider the restriction to any Levi subalgebra. We deduce results on decomposition numbers and branching coefficients of Schur algebras over fields of positive characteristic, generalizing those of Kleshchev [13] and of Tan and Teo [19].

* Corresponding author.
E-mail addresses: j.chuang@city.ac.uk (J. Chuang), tankm@nus.edu.sg (K.M. Tan).
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1. Introduction

The complete determination of the decomposition numbers of the symmetric groups and Schur algebras in positive characteristic p is a well-known and longstanding open problem, for which a complete solution does not seem to be forthcoming. Related to these decomposition numbers are the q-decomposition numbers arising from the canonical basis for the Fock space representation of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$. These q-decomposition numbers, as conjectured by Leclerc and Thibon [15] and shown by Varagnolo and Vasserot [20], are polynomials in q with nonnegative integer coefficients and when evaluated at $q = 1$ give the corresponding decomposition numbers for the v-Schur algebra in characteristic zero where v is a primitive n-th root of unity. As shown by James [9], when $n = p$, the decomposition matrix for the Schur algebra can be obtained by postmultiplying the decomposition matrix of the v-Schur algebra in characteristic zero by an adjustment matrix which has nonnegative integer entries and is unitriangular when the indexing set is suitably ordered. As such, the q-decomposition numbers provide a first approximation to the decomposition numbers of the Schur algebras.

James's Conjecture asserts that this first approximation is in fact an equality whenever the indexing partitions have p-weight less than p. Even though the conjecture is now known to be false in general [21], it has been proved in some cases, such as in Rouquier blocks [3] and in blocks with p-weight less than 5 [5,4]. It has also been shown that the first approximation is in fact an equality in many cases irrespective of the p-weight of the indexing partitions; see for example [18,19].

In [13], Kleshchev introduces the combinatorics of sign sequences, and uses it to describe the decomposition number $d_{\lambda\mu}$ when the partition λ is obtained from μ by moving one node. Subsequently, the present authors and Miyachi [2] provide closed formulas for the corresponding q-decomposition number $d_{\lambda\mu}(q)$ using the same combinatorics. More recently, the second author and Teo [19] provide closed formulas for $d_{\lambda\mu}(q)$ when λ is obtained from μ by moving any number of nodes as long as all of them have the same n-residue, and show that, when $n = p$, $d_{\lambda\mu}(1) = d_{\lambda\mu}$. The astute reader of [19] who is familiar with the work of Frenkel and Khovanov in [6] will be struck by the uncanny similarity between the last closed formulas and those describing the canonical bases of tensor powers $V_2^{\otimes d}$ of the natural two-dimensional representation V_2 of the quantum enveloping algebra $U_q(\mathfrak{sl}_2)$, although the former is formulated using the combinatorics of sign sequences while the latter is described by graphical calculus. It is natural to attempt to find out the exact relationship between the latter canonical bases with that of the Fock space representation. This is the main motivation of our work appearing in this paper.

We briefly describe our results here. For our purposes it suffices to consider the subalgebra $\mathbf{U} = U_q(\widehat{\mathfrak{sl}}_n)$ of $U_q(\widehat{\mathfrak{gl}}_n)$. Let $\mathbf{n} = (n_1, \dots, n_r)$ be a tuple of positive integers such that $n_1 + \cdots + n_r = n$, and let $\mathbf{U}(\mathbf{n})$ denote the subalgebra of $\mathbf{U}$ isomorphic to $U_q(\mathfrak{sl}_{n_1}) \otimes \cdots \otimes U_q(\mathfrak{sl}_{n_r})$. We show that the Fock space representation $\mathcal{F}_s$ indexed by an integer s (see subsection 2.3 for formal definition), when restricted from $\mathbf{U}$ to $\mathbf{U}(\mathbf{n})$,