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An inequality for the matrix pressure function and applications

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ABSTRACT

We prove an *a priori* lower bound for the pressure, or p-norm joint spectral radius, of a measure on the set of $d \times d$ real matrices which parallels a result of J. Bochi for the joint spectral radius. We apply this lower bound to give new proofs of the continuity of the affinity dimension of a self-affine set and of the continuity of the singular-value pressure for invertible matrices, both of which had been previously established by D.-J. Feng and P. Shmerkin using multiplicative ergodic theory and the subadditive variational principle. Unlike the previous proof, our lower bound yields algorithms to rigorously compute the pressure, singular value pressure and affinity dimension of a finite set of matrices to within an *a priori* prescribed accuracy in finitely many computational steps. We additionally deduce a related inequality for the singular value pressure for measures on the set of 2×2 real matrices, give a precise characterisation of the discontinuities of the singular value pressure function for two-dimensional matrices, and prove a general theorem relating the zero-temperature limit of the matrix pressure to the joint spectral radius.

1. Introduction

If $A_1, \ldots, A_N$ are $d \times d$ real matrices and $s > 0$ a real number, we may define the *(norm) pressure* of $A_1, \ldots, A_N$ to be the quantity

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$$\mathbf{M}((A_1,\ldots,A_N),s) := \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{i_1,\ldots,i_n=1}^{N}\|A_{i_1}\cdots A_{i_n}\|^s\right) \in [-\infty,+\infty),$$

the existence of the limit being guaranteed by subadditivity. This quantity has also been studied in the form of the p-norm joint spectral radius, or p-radius, defined by

$$\varrho_p(A_1,\ldots,A_N) := \lim_{n\to\infty}\left(\frac{1}{N^n}\sum_{i_1,\ldots,i_n=1}^{N}\|A_{i_1}\cdots A_{i_n}\|^p\right)^{\frac{1}{np}} = N^{-\frac{1}{p}}e^{\mathbf{M}((A_1,\ldots,A_n),p)/p} \tag{1}$$

where it is usually assumed that $p \geq 1$. (Here, and in general throughout the paper, we adopt the conventions $\log 0 := -\infty$, $e^{-\infty} := 0$.) The norm pressure and p-radius have been extensively investigated for their connections with wavelet analysis [25,30,46], the stability of switched linear systems [32], and thermodynamic formalism and multifractal analysis [16–18]; in recent years significant attention has been given to the efficient computation of the p-radius [27,28,33,34,36].

In this article we shall also be concerned with a related quantity, the *singular value pressure* of a finite set of matrices. Let $M_d(\mathbb{R})$ denote the vector space of all $d\times d$ real matrices, and let $\sigma_1(A),\ldots,\sigma_d(A)$ denote the singular values of a matrix $A \in M_d(\mathbb{R})$, which are defined to be the non-negative square roots of the eigenvalues of the positive semidefinite matrix A^*A, listed in decreasing order with repetition in the case of multiplicity. For each $s > 0$ and $A \in M_d(\mathbb{R})$ we define

$$\varphi^s(A) := \begin{cases} \sigma_1(A)\cdots\sigma_k(A)\sigma_{k+1}(A)^{s-k}, & k \leq s \leq k+1 \leq d \\ |\det A|^{\frac{s}{d}}, & s \geq d. \end{cases}$$

The function φ may easily be seen to be upper semi-continuous in (A,s) with discontinuities occurring precisely when s is an integer from 1 to $d-1$ such that $\sigma_{s+1}(A) = 0 < \sigma_s(A)$. We have $\varphi^s(AB) \leq \varphi^s(A)\varphi^s(B)$ for all $A,B \in M_d(\mathbb{R})$ and $s > 0$, see e.g. [11, Lemma 2.1]. For $A_1,\ldots,A_N \in M_d(\mathbb{R})$ and $s > 0$ we define the *singular value pressure* of $A_1,\ldots,A_N$ by

$$\mathbf{P}((A_1,\ldots,A_N),s) := \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{i_1,\ldots,i_n=1}^{N}\varphi^s\left(A_{i_1}\cdots A_{i_n}\right)\right) \in [-\infty,+\infty).$$

The singular value pressure plays a pivotal role in the dimension theory of self-affine fractals and has been extensively applied in that context (see e.g. [11,26,44]). Let us recall the definition of a self-affine set. If $T_1,\ldots,T_N\colon \mathbb{R}^d \to \mathbb{R}^d$ are contractions with respect to the Euclidean metric – that is, if there exists $\lambda \in [0,1)$ such that $\|T_ix - T_iy\| \leq \lambda\|x-y\|$ for all $x,y \in \mathbb{R}^d$ and $i = 1,\ldots,N$ – then by a well-known theorem of J. E. Hutchinson [24]