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Weak containment rigidity for distal actions ☆

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ABSTRACT

We prove that if a measure distal action α of a countable group Γ is weakly contained in a strongly ergodic probability measure preserving action β of Γ, then α is a factor of β. In particular, this applies when α is a compact action.

As a consequence, we show that the weak equivalence class of any strongly ergodic action completely remembers the weak isomorphism class of the maximal distal factor arising in the Furstenberg–Zimmer Structure Theorem.

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1. Introduction

The notion of weak containment for group actions was introduced by A. Kechris [16] as an analogue of the notion of weak containment for unitary representations. Let $\Gamma \curvearrowright^{\alpha} (X, \mu)$ and $\Gamma \curvearrowright^{\beta} (Y, \nu)$ be two probability measure preserving (p.m.p.) actions of a countable group Γ. Then α is said to be *weakly contained* in β (in symbols, $\alpha \prec \beta$) if for any finite set $S \subset \Gamma$, finite measurable partition $\{A_i\}_{i=1}^{n}$ of X, and $\varepsilon > 0$, we can find a measurable partition $\{B_i\}_{i=1}^{n}$ of Y such that for all $\gamma \in S$ and $i, j \in \{1, ..., n\}$ we have

$$|\mu(\gamma A_i \cap A_j) - \nu(\gamma B_i \cap B_j)| < \varepsilon.$$

If $\alpha \prec \beta$ and $\beta \prec \alpha$, we say that α is *weakly equivalent* to β.

We say that α is a *factor* of β, or that β is an *extension* of α, if there exists a measurable, measure preserving map $\theta : Y \to X$ such that $\theta(\gamma y) = \gamma\theta(y)$, for all $\gamma \in \Gamma$ and almost every $y \in Y$. The map θ is called a *factor map* or an *extension*. If in addition there is a conull set $Y_0 \subseteq Y$ such that θ is one-to-one on Y_0, then θ is called an *isomorphism* and we say that α is *isomorphic* to β. The actions α and β are said to be *weakly isomorphic* if each is a factor of the other.

As the terminology suggests, if α is a factor of β, then α is weakly contained in β. The main goal of this note is to establish a rigidity result which provides a general instance when the converse holds.

Theorem 1.1. *Let Γ be a countable group, $\Gamma \curvearrowright^{\alpha} (X, \mu)$ be a measure distal p.m.p. action, and $\Gamma \curvearrowright^{\beta} (Y, \nu)$ be a strongly ergodic p.m.p. action.*

If α is weakly contained in β, then α is a factor of β. In particular, if a compact action α is weakly contained in a strongly ergodic action β, then α is a factor of β.

Before recalling the notions involved in Theorem 1.1, let us put it into context and outline its proof.

Weak containment and weak equivalence have received much attention since their introduction. In [16], A. Kechris shows that cost varies monotonically with weak containment and in [17] Kechris uses this monotonicity to obtain a new proof that free groups have fixed price. Several other measurable combinatorial parameters of actions are known to respect weak containment and hence are invariants of weak equivalence; see [1,10,11].

In [3], M. Abért and B. Weiss exhibit a remarkable anti-rigidity phenomenon for weak containment by showing that every free p.m.p. action of Γ weakly contains the Bernoulli action over an atomless base space. The Abért–Weiss Theorem was extended in [21] and used to show that every weak equivalence class contains “unclassifiably many” isomorphism classes of actions, thus ruling out the possibility of weak equivalence superrigidity. These anti-rigidity results stand in marked contrast to the rigidity exhibited in Theo-