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A construction of non-Kähler Calabi–Yau manifolds and new solutions to the Strominger system

Teng Fei

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ABSTRACT

We propose a new construction of compact non-Kähler Calabi–Yau manifolds with balanced metrics and study the Strominger system on them. In particular, we obtain explicit solutions to the Strominger system with degeneracies on $\Sigma_g \times T^4$, where Σ_g is an immersed minimal surface of genus $g \geq 3$ in flat T^3.

1. Introduction

We call a compact complex manifold a Calabi–Yau, to be Kähler or not, if its canonical bundle is holomorphically trivial. Calabi–Yau manifolds, especially Calabi–Yau 3-folds, have been extensively studied ever since Yau's solution to the Calabi conjecture [48,50]. Among many other enthralling problems, motivated from either geometry or string theory, there stands out the moduli problem. There exists a vast collection of literature devoted to understanding the moduli space of Calabi–Yau manifolds. We refer to the

E-mail address: tfei@math.mit.edu.

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beautiful survey paper [51] and the references therein for more details. In our context, we would like to single out what is usually called the "Reid's fantasy". In [39], Reid made the wild conjecture that all the reasonably nice compact 3-folds with trivial canonical bundle (including non-Kähler ones necessarily) can be connected with each other via conifold transitions. Reid's idea was supported by the work of Candelas–de la Ossa [7], where explicit Ricci-flat Kähler metrics are found on both deformed and resolved conifolds and their limiting behaviors are analyzed.

In order to globalize Candelas–de la Ossa's result, it is natural to ask the question that how canonical we can choose Hermitian metrics on compact non-Kähler Calabi–Yau manifolds. The first way to deal with this problem is to understand how to generalize the Calabi–Yau theorem in the non-Kähler setting. In particular, the balanced version of Gauduchon conjecture (for example see [45, Conjecture 4.1 and 4.2]) remains to be solved. A consequence of this conjecture is that on complex balanced manifolds with trivial canonical bundle one can always find a balanced metric within a suitable cohomology class such that its first Bott–Chern form vanishes. Progress was made by Székelyhidi–Tosatti–Weinkove [41] in this direction.

A second approach to this problem is to solve the Strominger system. This is a system of PDEs proposed by Strominger [40] in the study of heterotic strings with torsion which we shall formulate. Let (X^3, g, J) be a Hermitian 3-fold (not necessarily Kähler) with holomorphically trivial canonical bundle and let Ω be a nowhere-vanishing holomorphic $(3, 0)$-form on X. We denote the positive $(1, 1)$-form associated with g by ω and the curvature form of $(T_{\mathbb{C}}X, g)$ by R. In addition, let (E, h) be a holomorphic vector bundle with metric over X and let F be its curvature form. The Strominger system consists of the following equations:

$$F \wedge \omega^2 = 0, \quad F^{0,2} = F^{2,0} = 0, \tag{1}$$

$$\sqrt{-1}\partial\overline{\partial}\omega = \frac{\alpha'}{4}(\mathrm{Tr}(R \wedge R) - \mathrm{Tr}(F \wedge F)), \tag{2}$$

$$\mathrm{d}(\|\Omega\|_\omega \cdot \omega^2) = 0. \tag{3}$$

Equations (1), (2) and (3) are known as the Hermitian–Yang–Mills equation, the anomaly cancellation equation and the conformally balanced equation respectively.

If we assume that ω is Kähler and take $E = T_{\mathbb{C}}X$ with $h = g$, the anomaly cancellation (2) is automatic. It follows that the whole system is reduced to an equation requiring g to be Ricci-flat. In this sense, the Strominger system generalizes the complex Monge–Ampère equation used in Kähler geometry. Therefore we may regard solutions to Strominger system as canonical metrics, even on a non-Kähler background. Actually, the first method mentioned above can be incorporated into this picture. It is well-known that for X with trivial canonical bundle, the conformally balanced condition (3) is equivalent to that the restricted holonomy of X with respect to the Strominger–Bismut connection is contained in $SU(3)$, see [40, Section 2] and [33, Lemma 3.1]. However, this is very far from requiring Ω to be parallel under the Strominger–Bismut connection. It turns out