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Multivariate orthogonal polynomials and integrable systems

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ABSTRACT

Multivariate orthogonal polynomials in D real dimensions are considered from the perspective of the Cholesky factorization of a moment matrix. The approach allows for the construction of corresponding multivariate orthogonal polynomials, associated second kind functions, Jacobi type matrices and associated three term relations and also Christoffel–Darboux formulae. The multivariate orthogonal polynomials, their second kind functions and the corresponding Christoffel–Darboux kernels are shown to be quasi-determinants—as well as Schur complements—of bordered truncations of the moment matrix; quasi-tau functions are introduced. It is proven that the second kind functions are multivariate Cauchy transforms of the multivariate orthogonal polynomials. Discrete and continuous deformations of the measure lead to Toda type integrable hierarchy, being the corresponding flows described through Lax and Zakharov–Shabat equations; bilinear equations are found. Varying size matrix nonlinear partial difference and differential equations of the 2D Toda lattice type are shown to be solved by matrix coefficients of the multivariate orthogonal polynomials. The discrete flows, which are shown to be connected with a Gauss–Borel factorization of the Jacobi type matrices and its quasi-determinants, lead to expressions for

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KP equations

the multivariate orthogonal polynomials and their second kind functions in terms of shifted quasi-tau matrices, which generalize to the multidimensional realm, those that relate the Baker and adjoint Baker functions to ratios of Miwa shifted τ-functions in the 1D scenario. In this context, the multivariate extension of the elementary Darboux transformation is given in terms of quasi-determinants of matrices built up by the evaluation, at a poised set of nodes lying in an appropriate hyperplane in $\mathbb{R}^D$, of the multivariate orthogonal polynomials. The multivariate Christoffel formula for the iteration of m elementary Darboux transformations is given as a quasi-determinant. It is shown, using congruences in the space of semi-infinite matrices, that the discrete and continuous flows are intimately connected and determine nonlinear partial difference–differential equations that involve only one site in the integrable lattice behaving as a Kadomstev–Petviashvili type system. Finally, a brief discussion of measures with a particular linear isometry invariance and some of its consequences for the corresponding multivariate polynomials is given. In particular, it is shown that the Toda times that preserve the invariance condition lay in a secant variety of the Veronese variety of the fixed point set of the linear isometry.

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