

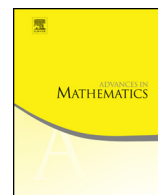


ELSEVIER

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim



Multivariate orthogonal polynomials and integrable systems

Gerardo Ariznabarreta^{1,2}, Manuel Mañas^{*,1}

*Departamento de Física Teórica II (Métodos Matemáticos de la Física),
Universidad Complutense de Madrid, Ciudad Universitaria, Plaza de Ciencias 1,
28040 Madrid, Spain*

ARTICLE INFO

Article history:

Received 26 September 2014

Accepted 25 June 2016

Available online xxxx

Communicated by Charles Fefferman

MSC:

15A23

33C45

37K10

37L60

42C05

46L55

Keywords:

Multivariate orthogonal polynomials

Borel–Gauss factorization

Quasi-determinants

Christoffel–Darboux kernels

Darboux transformations

Christoffel formula

Quasi-tau matrices

Kernel polynomials

Integrable hierarchies

Toda equations

ABSTRACT

Multivariate orthogonal polynomials in D real dimensions are considered from the perspective of the Cholesky factorization of a moment matrix. The approach allows for the construction of corresponding multivariate orthogonal polynomials, associated second kind functions, Jacobi type matrices and associated three term relations and also Christoffel–Darboux formulae. The multivariate orthogonal polynomials, their second kind functions and the corresponding Christoffel–Darboux kernels are shown to be quasi-determinants—as well as Schur complements—of bordered truncations of the moment matrix; quasi-tau functions are introduced. It is proven that the second kind functions are multivariate Cauchy transforms of the multivariate orthogonal polynomials. Discrete and continuous deformations of the measure lead to Toda type integrable hierarchy, being the corresponding flows described through Lax and Zakharov–Shabat equations; bilinear equations are found. Varying size matrix nonlinear partial difference and differential equations of the 2D Toda lattice type are shown to be solved by matrix coefficients of the multivariate orthogonal polynomials. The discrete flows, which are shown to be connected with a Gauss–Borel factorization of the Jacobi type matrices and its quasi-determinants, lead to expressions for

* Corresponding author.

E-mail addresses: gariznab@ucm.es (G. Ariznabarreta), manuel.manas@ucm.es (M. Mañas).

¹ Thanks for financial support from the Spanish Ministerio de Economía y Competitividad research project [MTM2015-65888-C4-3-P], *Ortogonalidad, teoría de la aproximación y aplicaciones en física matemática*.

² Thanks for financial support from the Universidad Complutense de Madrid Program “Ayudas para Becas y Contratos Complutenses Predoctorales en España 2011”.

KP equations

the multivariate orthogonal polynomials and their second kind functions in terms of shifted quasi-tau matrices, which generalize to the multidimensional realm, those that relate the Baker and adjoint Baker functions to ratios of Miwa shifted τ -functions in the 1D scenario. In this context, the multivariate extension of the elementary Darboux transformation is given in terms of quasi-determinants of matrices built up by the evaluation, at a poised set of nodes lying in an appropriate hyperplane in $\mathbb{R}^D$, of the multivariate orthogonal polynomials. The multivariate Christoffel formula for the iteration of m elementary Darboux transformations is given as a quasi-determinant. It is shown, using congruences in the space of semi-infinite matrices, that the discrete and continuous flows are intimately connected and determine nonlinear partial difference–differential equations that involve only one site in the integrable lattice behaving as a Kadomstev–Petviashvili type system. Finally, a brief discussion of measures with a particular linear isometry invariance and some of its consequences for the corresponding multivariate polynomials is given. In particular, it is shown that the Toda times that preserve the invariance condition lay in a secant variety of the Veronese variety of the fixed point set of the linear isometry.

© 2016 Elsevier Inc. All rights reserved.

Contents

1.	Introduction	630
1.1.	On multivariate orthogonal polynomials	630
1.2.	On the Toda equations	632
1.3.	Gauss–Borel factorization in integrable systems and orthogonal polynomials	634
1.4.	On quasi-determinants	635
1.5.	Aims, results and structure of the paper	636
2.	Multivariate orthogonality à la Cholesky	639
2.1.	Ordering the monomials	639
2.2.	Monomials and symmetric tensor powers	640
2.3.	Cholesky factorization of the moment matrix	641
2.4.	MVOPR	644
2.5.	Functions of the second kind	647
2.6.	The shift matrices	650
2.7.	Jacobi matrices and three term relations	652
2.8.	Christoffel–Darboux formulae	654
3.	On discrete Toda and MVOPR	657
3.1.	The discrete flows	657
3.2.	Quasi-tau functions formulae for MVOPR	663
3.3.	Transforming the Christoffel–Darboux kernels and kernel polynomials	667
3.4.	Elementary Darboux transformations and the sample matrix trick	668
3.5.	Multivariate Christoffel formula and quasi-determinants	672
4.	On continuous Toda and MVOPR	682
4.1.	The continuous flows	682
4.2.	Baker functions. Lax and Zakharov–Shabat equations	685
4.3.	Miwa shifts and discrete flows	687
4.4.	Bilinear equations	688
4.5.	Toda type integrable equations	690
5.	KP type equations via congruences	693
5.1.	The congruence technique	693

Download English Version:

<https://daneshyari.com/en/article/6425143>

Download Persian Version:

<https://daneshyari.com/article/6425143>

[Daneshyari.com](https://daneshyari.com)