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# Completeness and spectral synthesis of nonselfadjoint one-dimensional perturbations of selfadjoint operators ☆

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A B S T R A C T

We study spectral properties of nonselfadjoint rank one perturbations of compact selfadjoint operators. The problems under consideration include completeness of eigenvectors, relations between completeness of the perturbed operator and its adjoint, and the spectral synthesis problem. We obtain new criteria for completeness and spectral synthesis in this class as well as a series of counterexamples which show that the spectral structure of rank one perturbations is, in general, unexpectedly rich and complicated.
A parallel spectral theory is developed for one-dimensional singular perturbations of unbounded selfadjoint operators. Our approach is based on a functional model for this class which translates the properties of operators to completeness

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problems for systems of reproducing kernels and their biorthogonals in some spaces of analytic (entire) functions.


## 1. Introduction and main results

A Banach or Hilbert space linear operator will be called *complete* if it has a complete set of eigenvectors and root vectors. Despite a large effort devoted to the study of criteria of completeness of operators, our understanding still is far from perfect, even for the case of ordinary differential operators.

### 1.1. Theorems of Keldyš and Macaev

Most general abstract completeness results are due to Keldyš [46,47] and Macaev [60] (see, also, [33, Chapter V]). We recall that an operator $S$ on a Hilbert space belongs to the Macaev ideal $\mathfrak{S}_\omega$ if it is compact and its singular numbers $s_k$ satisfy the relation $\sum_{k\ge1} k^{-1}s_k < \infty$.

**Theorem** *(Keldyš, 1951). Suppose $\mathcal{A}$ is a selfadjoint Hilbert space operator that belongs to a Schatten ideal $\mathfrak{S}_p$, $0 < p < \infty$ and satisfies $\ker\mathcal{A} = 0$. Let $\mathcal{L} = \mathcal{A}(I + S)$, where $S$ is compact and $\ker(I + S) = 0$. Then the operators $\mathcal{L}$ and $\mathcal{L}^*$ are complete.*

**Theorem** *(Macaev, 1961). If $\mathcal{L} = \mathcal{A}(I + S)$, where $\mathcal{A}$, $S$ are compact operators on a Hilbert space, $\mathcal{A}$ is selfadjoint, $S \in \mathfrak{S}_\omega$ and $\ker\mathcal{A} = \ker(I + S) = 0$, then $\mathcal{L}$ and $\mathcal{L}^*$ are complete.*

A proof of this theorem and its generalizations to operator pencils can be found in [62]. As Macaev proved in [61, Theorem 3], in the above result $\mathfrak{S}_\omega$ can be replaced by a wider class, depending on $\mathcal{A}$. Perturbations of a selfadjoint compact operator $\mathcal{A}$ that have the form $\mathcal{A}(I + S)$ or $(I + S)\mathcal{A}$, where $S$ is compact, are called *weak perturbations*. In [64], Macaev and Mogul'skii give an explicit condition on the spectrum of $\mathcal{A}$, equivalent to the property that all weak perturbations of $\mathcal{A}$ are complete; see also [63].

### 1.2. Statement of the problems

In this article, we consider rank $n$ perturbations of compact selfadjoint operators, which are neither weak nor dissipative. Our main results are concerned with rank one (nondissipative) perturbations of compact selfadjoint operators. We study the following spectral properties of the operators from this class:

- Completeness of eigenvectors or root vectors;