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Critical measures for vector energy: Global structure of trajectories of quadratic differentials

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A B S T R A C T

Saddle points of a vector logarithmic energy with a vector polynomial external field on the plane constitute the *vector-valued critical measures*, a notion that finds a natural motivation in several branches of analysis. We study in depth the case of measures $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$ when the mutual interaction comprises both attracting and repelling forces.

For arbitrary vector polynomial external fields we establish general structural results about critical measures, such as their characterization in terms of an algebraic equation solved by an appropriate combination of their Cauchy transforms, and the symmetry properties (or the S-properties) exhibited by such measures. In consequence, we conclude that vector-valued critical measures are supported on a finite number of analytic arcs, that are trajectories of a quadratic differential globally defined on a three-sheeted Riemann surface. The complete description of the so-called critical graph for such a differential is the key to the construction of the critical measures.

We illustrate these connections studying in depth a one-parameter family of critical measures under the action of a cubic external field. This choice is motivated by the asymptotic analysis of a family of (non-hermitian) multiple orthogonal polynomials, that is subject of a forthcoming paper. Here we compute explicitly the Riemann surface and the corresponding quadratic differential, and analyze the dynamics of its critical graph as a function of the

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parameter, giving a detailed description of the occurring phase transitions. When projected back to the complex plane, this construction gives us the complete family of vector-valued critical measures, that in this context turn out to be vector-valued equilibrium measures.

1. Introduction and statement of main results

1.1. Historical background

Statistical systems of many particles have been object of intense analysis for a long time, both from the perspective of physics and mathematics. The study of a particular type of interacting particle systems, the so-called determinantal point processes (and their cousins, Pfaffian processes), has been especially fruitful in the past thirty years. This success can be explained both by the ubiquitous character and flexibility of these models (describing the eigenvalues of several random matrix ensembles, non-intersecting diffusion paths, random growth models, random tilings, to mention a few) and the introduction of new tools, intrinsically related with the analytic theory of orthogonal polynomials and their generalizations.

A common feature of these models is that either the joint probability density, the correlation functions, the normalization constant or the generating function can be expressed as a determinant (resp., a Pfaffian), and the right selection of the functions appearing in these determinants unveils the integrable structure of the underlying processes. Another unified property of these models is the possibility to put them in the framework of the so-called log gases, where the particles behave as charges on one or two dimensional sets, subject to the logarithmic interaction.

The best known case is the spectrum of some unitary matrix ensembles [28,29], described in terms of classical families of orthogonal polynomials on the real line. Their zeros, all real and simple, satisfy an electrostatic model that goes all the way back to Stieltjes [76], and solve a minimization problem for the associated (logarithmic) energy. In other words, the zeros provide an equilibrium configuration on the conducting real line. Similar situation occurs for some non-intersecting paths models or for the six-vertex model in statistical mechanics [24], to mention some more examples.

Another classical framework for orthogonal polynomials (orthogonality on the unit circle), developed in the seminal works of Szegő [78], is connected to the analysis of the Ising model [30,31].

Further immediate generalizations of the problems above oblige us to leave the real line and extend the notion of orthogonality and the associated log gas models to the complex plane. The so-called *non-hermitian orthogonal polynomials* appear naturally in