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Non-abelian Littlewood–Offord inequalities

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ABSTRACT

In 1943, Littlewood and Offord proved the first anti-concentration result for sums of independent random variables. Their result has since then been strengthened and generalized by generations of researchers, with applications in several areas of mathematics.

In this paper, we present the first non-abelian analogue of the Littlewood–Offord result, a sharp anti-concentration inequality for products of independent random variables.

1. Introduction

In 1943, motivated by their studies of random polynomials, Littlewood and Offord [20] proved a remarkable fact about the distribution of a sum of independent random vari-

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ables. Let V be a sequence of (not necessarily different) non-zero real numbers $a_1, \dots, a_n$ and set

$$\rho_V := \sup_{b \in \mathbb{R}} \mathbf{P}(\sum_{i=1}^{n} \hat{a}_i = b),$$

where the $\hat{a}_i$ are independent random variables taking values $\pm a_i$ with probability 1/2.

Theorem 1.1.

$$\rho_V = O(n^{-1/2} \log n).$$

Here and hereafter, the asymptotic notation is used under the assumption that $n \to \infty$. Soon after their paper, Erdős [3], removing the $\log n$ term, optimized the bound.

Theorem 1.2 *(Littlewood–Offord–Erdős).*

$$\rho_V \leq \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n} = O(n^{-1/2}).$$

The bound is sharp, as shown by taking all $a_i = 1$. It is easy to see that in this case $\mathbf{P}(\sum_{i=1}^n \hat{a}_i = \delta) = \frac{\binom{n}{\lfloor n/2 \rfloor}}{2^n}$, where $\delta = 1$ if n is odd and 0 otherwise. In [13], Kleitman generalized Theorem 1.2 to complex setting.

It is best to relate Theorem 1.2 to the classical Berry–Esseen theorem, which asserts that if the a_i are all of magnitude 1, then the distribution of $\frac{1}{\sqrt{n}} \sum_i^n \hat{a}_i$ converges to the normal distribution with rate $O(n^{-1/2})$. This implies that for any point b, $\mathbf{P}(\sum_{i=1}^n \hat{a}_i = b) = O(n^{-1/2})$. Theorem 1.2 strengthens this fact significantly, asserting that the probability in question is always $O(n^{-1/2})$, *regardless of the magnitude of the* a_i.

Theorem 1.2 has become the starting point of a long line of research, which continues through several decades and has recently become very active (see, for instance, the survey [24]). It has been strengthened (under various conditions) and generalized in different directions by many researchers, including Esseen, Kolmogorov, Rogozin, Halász, Stanley, Kleitman, Szemerédi–Sárközy, Tao, and others; see, for instance [4–6,10,11,13,14,12,15,16,21,23,25,27,26,28,30–32]. These results are often referred to as anti-concentration inequalities, and have found surprising applications in different areas of mathematics, including random matrix theory. For more details, the reader may want to check the recent survey [24].

A limitation to all existing extensions of Theorem 1.2 is that they only apply for random variables taking values in an abelian group (in most cases $\mathbb{R}^d$ or $\mathbb{C}$), as the available proof techniques only work in this setting.

The goal of this paper is to initiate the study of the anti-concentration phenomenon in the non-abelian setting. Let V be a sequence of (not necessarily distinct) non-trivial