Advances in Mathematics 299 (2016) 174–201

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Flow by powers of the Gauss curvature ☆

Ben Andrews [a], Pengfei Guan [b,*], Lei Ni [c,*]

[a] *Mathematical Sciences Institute, Australia National University, ACT 2601, Australia*
[b] *Department of Mathematics and Statistics, McGill University, Montreal, Quebec, H3A 2K6, Canada*
[c] *Department of Mathematics, University of California at San Diego, La Jolla, CA 92093, USA*

A R T I C L E I N F O

Article history:
Received 2 October 2015
Received in revised form 10 March 2016
Accepted 13 May 2016
Available online xxxx
Communicated by Erwin Lutwak

MSC:
35K55
35B65
53A05
58G11

Keywords:
Entropy
Gauss Curvature
Monotonicity
Regularity Estimates
Curvature Image
Entropy Stability Estimates

A B S T R A C T

We prove that convex hypersurfaces in $\mathbb{R}^{n+1}$ contracting under the flow by any power $\alpha > \frac{1}{n+2}$ of the Gauss curvature converge (after rescaling to fixed volume) to a limit which is a smooth, uniformly convex self-similar contracting solution of the flow. Under additional central symmetry of the initial body we prove that the limit is the round sphere for $\alpha \geq 1$.

☆ The first author was supported by Discovery Projects grant DP120102462 and Australian Laureate Fellowship FL150100126 of the Australian Research Council. The research of the second author is partially supported by an NSERC Discovery Grant, and the research of the third author is partially supported by NSF grant DMS-1401500.
* Corresponding authors.
E-mail addresses: Ben.Andrews@anu.edu.au (B. Andrews), guan@math.mcgill.ca (P. Guan), lni@math.ucsd.edu (L. Ni).

http://dx.doi.org/10.1016/j.aim.2016.05.008

1. Introduction

In this paper we study the flow of convex hypersurfaces $\tilde{X}(\cdot,\tau) : M \to \mathbb{R}^{n+1}$ by the α-power of Gauss curvature:

$$\frac{\partial}{\partial \tau}\tilde{X}(x,\tau) = -\tilde{K}^{\alpha}(x,\tau)\,\nu(x,\tau). \tag{1.1}$$

Here $\nu(x,\tau)$ is the unit exterior normal at $\tilde{X}(x,\tau)$ of $\tilde{M}_\tau = \tilde{X}(M,\tau)$, and $\tilde{K}(x,\tau)$ is the Gauss curvature of $\tilde{M}_\tau$ at $\tilde{X}(x,\tau)$ (the tildes distinguish these from the normalized counterparts introduced below).

Equation (1.1) is a parabolic fully nonlinear equation of Monge–Ampére type, hence the study sheds light on the general theory of such equations. The case $\alpha = 1$ was proposed by Firey [19] as a model for the wearing of tumbling stones. The equation with general powers also arises in the study of affine geometry and of image analysis [1,15,31,33,34]. For large α the equation becomes more degenerate and for small α it becomes more singular. Studying them together gives an example of nonlinear parabolic equations with varying degeneracy. The interested reader may consult [7] for motivation for the study of this flow. For the short time existence, it was proved in [39] for $\alpha = 1$, and for any $\alpha > 0$ in [17] that the flow shrinks any smooth, uniformly convex body $M_0 = \partial\Omega_0$ to a point z_∞ in finite time $T > 0$. An important differential Harnack estimate (also referred as Li–Yau–Hamilton type estimate) was later proved in [18] (see also [2]). The current paper concerns the asymptotics of the solutions as the time approaches to the singular time T.

The study of the asymptotic behavior is equivalent to the large time behavior of the normalized flow, which is obtained by re-scaling about the final point to keep the enclosed volume fixed, and suitably re-parameterizing the time variable (see section 3 for details):

$$\frac{\partial}{\partial t}X(x,t) = -\frac{K^{\alpha}(x,t)}{\fint_{\mathbb{S}^n} K^{\alpha-1}}\,\nu(x,t) + X(x,t). \tag{1.2}$$

Here we write $\fint_{\mathbb{S}^n} f(x)d\theta(x) = \frac{1}{\omega_n}\int_{\mathbb{S}^n} f(x)\,d\theta(x)$ for any continuous function f on $\mathbb{S}^n$, where $d\theta(x)$ is the spherical Lebesgue measure and $\omega_n = |\mathbb{S}^n|$, and we interpret K as a function on $\mathbb{S}^n$ via the Gauss map diffeomorphism $\nu:\ M_t \to \mathbb{S}^n$. It can be easily checked that $M_t = X(M,t)$ encloses a convex body Ω_t whose volume $|\Omega_t|$ changes according to the equation:

$$\begin{aligned}\frac{d}{dt}|\Omega_t| &= -\frac{1}{\fint_{\mathbb{S}^n} K^{\alpha-1}}\int_{M_t} K^{\alpha} + \int_{M_t}\langle X,\nu\rangle \\ &= -\omega_n + (n+1)|\Omega_t|.\end{aligned}$$

Hence if $|\Omega_0| = |B(1)| = \frac{\omega_n}{n+1}$, where $B(1) \subset \mathbb{R}^{n+1}$ is the unit ball, then $|\Omega_t| = |B(1)|$ for all t.