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Semi-classical weights and equivariant spectral theory

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ABSTRACT

We prove inverse spectral results for differential operators on manifolds and orbifolds invariant under a torus action. These inverse spectral results involve the asymptotic equivariant spectrum, which is the spectrum itself together with "very large" weights of the torus action on eigenspaces. More precisely, we show that the asymptotic equivariant spectrum of the Laplace operator of any toric metric on a generic toric orbifold determines the equivariant biholomorphism class of the orbifold; we also show that the asymptotic equivariant spectrum of a $\mathbb{T}^n$-invariant Schrödinger operator on $\mathbb{R}^n$ determines its potential in some suitably convex cases. In addition, we prove that the asymptotic equivariant spectrum of an S^1-invariant metric on S^2 determines the metric itself in many cases. Finally, we obtain an asymptotic equivariant inverse spectral result for weighted projective spaces. As a crucial ingredient in these inverse results, we derive a surprisingly simple formula for the asymptotic equivariant

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trace of a family of semi-classical differential operators invariant under a torus action.

1. Introduction

To what extent are the geometric properties of a Riemannian manifold determined by the spectrum of its Laplacian? In [11] the authors study a variation of this classical problem in the context of an isometric group action. This gives rise to the notion of *equivariant spectrum*, which is simply the spectrum together with the weights of the representations of the isometric group action on the eigenspaces. A classical tool used to answer inverse spectral questions is an asymptotic expansion for the heat kernel or the wave trace for the Laplacian. With this in mind, it is natural to try to use such an expansion for the equivariant heat kernel or the equivariant wave trace, thereby "counting" equivariant eigenfunctions.

In [10] the authors obtain asymptotic expansions for traces of operators on manifolds in the presence of an isometry. The goal of the present paper is to develop the techniques in [10] to obtain an asymptotic expansion for the equivariant trace of a semi-classical differential operator invariant under a group action; the equivariant trace encodes information about equivariant eigenspaces. One of our key observations is that by considering "very large" or *semi-classical* weights, which give rise to the asymptotic equivariant spectrum, it is possible to obtain a relatively simple formula for the asymptotic behavior of this equivariant trace. We now give a rough formulation of this result; a more precise version can be found in Theorem 5.1.

Consider a Riemannian manifold X^n admitting an isometric action of a torus $\mathbb{T}^m$. This action lifts to a Hamiltonian action on T^*X with moment map $\Phi : T^*X \to \mathbb{R}^m$. For generic $\alpha \in \mathbb{R}^m$, let $(T^*X)_\alpha := \Phi^{-1}(\alpha)/\mathbb{T}^m$ denote the symplectic reduction of T^*X at level α, and let χ_α be a character of $\mathbb{T}^m$ associated with α. Let $\mathcal{P}_h$ be a family of semi-classical differential operators that are $\mathbb{T}^m$-invariant. In applications we will consider $\mathcal{P}_h = h^2\triangle$ or $\mathcal{P}_h = h^2\triangle + V$. We denote by $\mathfrak{p}_0 : T^*X \to \mathbb{R}$ the leading symbol of $\mathcal{P}_h$. This induces a map $\mathfrak{p}_\alpha : (T^*X)_\alpha \to \mathbb{R}$. Let $\lambda_i(\alpha, h)$ for $i = 1, 2, \dots$ denote the $\frac{\alpha}{h}$-equivariant eigenvalues counted with multiplicities, that is, the eigenvalues of $\mathcal{P}_h$ restricted to $C^\infty(X)^{\frac{\alpha}{h}}$, where $C^\infty(X)^{\frac{\alpha}{h}}$ is the space of functions

$$\mathcal{C}^\infty(X)^{\frac{\alpha}{h}} = \{f \in \mathcal{C}^\infty(X) : f(gx) = \chi_{\frac{\alpha}{h}}(g)f(x)\}.$$

Consider the spectral measure

$$\mu_{\frac{\alpha}{h}}(\rho) := \sum_i \rho(\lambda_i(\alpha, h)), \tag{1}$$