

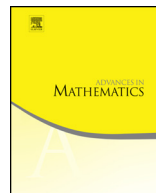


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# Real groups and Sylow 2-subgroups ☆☆☆☆☆



Gabriel Navarro<sup>a</sup>, Pham Huu Tiep<sup>b,\*</sup>

<sup>a</sup> *Departament d'Àlgebra, Universitat de València, 46100 Burjassot, València, Spain*

<sup>b</sup> *Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA*

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## ABSTRACT

If  $G$  is a finite real group and  $P \in \text{Syl}_2(G)$ , then  $P/P'$  is elementary abelian. This confirms a conjecture of Roderick Gow. In fact, we prove a much stronger result that implies Gow's conjecture.

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\* Corresponding author.

E-mail addresses: [gabriel.navarro@uv.es](mailto:gabriel.navarro@uv.es) (G. Navarro), [tiep@math.arizona.edu](mailto:tiep@math.arizona.edu) (P.H. Tiep).

## 1. Introduction

Reality questions on finite groups are almost as old as finite group theory itself, and they trace back to the work of Frobenius, Schur and Burnside.

An element  $g$  of a group  $G$  is **real** if  $g$  is conjugate to its inverse, and a character  $\chi$  of  $G$  is **real-valued** if  $\chi$  only takes real values. It is well-known that the number of real-valued irreducible complex characters of a finite group  $G$  is the number of conjugacy classes of  $G$  consisting of real elements, and a group  $G$  is **real** if all of its elements are real.

R. Gow's conjecture asserts that if  $G$  is a finite real group, then  $P/P'$  is an elementary abelian 2-group, where  $P$  is a Sylow 2-subgroup of  $G$ , and  $P'$  is the derived subgroup of  $P$ . This conjecture follows the main philosophy in finite group representation theory: global implies local or vice-versa.

In this paper we prove Gow's conjecture. The key is that we have found a more general statement with good inductive conditions which holds true (see [Theorem 2.11](#) below). This statement has interest on its own, and in its simplest case already implies a much stronger version of Gow's conjecture.

**Theorem A.** *Let  $G$  be a finite group with a Sylow 2-subgroup  $P$ . If all the odd-degree irreducible characters of  $G$  are real-valued, then  $P/P'$  is elementary abelian.*

Our proof of [Theorem A](#) uses the Classification of Finite Simple Groups; in fact, we are able to reduce it to a problem on almost simple groups. The converse of [Theorem A](#) is not true, however, even for solvable groups.

There is no block version of [Theorem A](#), except for the principal block. It is simply not true that if all the height zero characters in a 2-block  $B$  with defect group  $P$  are real-valued then  $P/P'$  is elementary abelian:  $D_{24}$ , for instance, is already a counterexample. On the other hand, we have the following non-trivial extension of [Theorem A](#).

**Theorem B.** *Let  $G$  be a finite group with a Sylow 2-subgroup  $P$ . If all the odd-degree irreducible characters in the principal 2-block of  $G$  are real-valued, then  $P/P'$  is elementary abelian.*

It is not true that if  $G$  is a real group, then  $P \in \text{Syl}_2(G)$  is real (a statement that would imply Gow's conjecture). The smallest counterexample is the group  $A_4 : Q_8$ .

[Theorems A and B](#) lead to an interesting problem: is it possible to give a group characterization of when all the odd-degree irreducible characters of  $G$  are real-valued? For solvable groups, we can show that this is the case if and only if  $P = N_G(P)$  and  $P' = \Phi(P)$ , the Frattini subgroup of  $P$  (i.e. the intersection of all maximal subgroups of  $P$ ), see [Theorem 2.13](#) below. But this is not true outside solvable groups, as  $G = A_5$  or  $G = A_9$  show us (in both directions). We also note that the converse of [Theorem B](#) is false, as shown by the example of  $SL_3(2)$ .

Finally, we have some evidence that there might be a version of Gow's conjecture for odd primes, but we will leave this problem for another place [\[25\]](#).

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