Advances in Mathematics 299 (2016) 331–360

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Real groups and Sylow 2-subgroups[☆,☆☆,☆☆☆]

Gabriel Navarro[a], Pham Huu Tiep[b,*]

[a] Departament d'Àlgebra, Universitat de València, 46100 Burjassot, València, Spain
[b] Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

ARTICLE INFO

Article history:
Received 18 December 2015
Received in revised form 16 May 2016
Accepted 17 May 2016
Available online 26 May 2016
Communicated by Henning Krause

MSC:
primary 20C15
secondary 20C33, 20D06, 20D20

Keywords:
Real groups
Real characters
Sylow 2-subgroups

ABSTRACT

If G is a finite real group and $P \in \mathrm{Syl}_2(G)$, then P/P' is elementary abelian. This confirms a conjecture of Roderick Gow. In fact, we prove a much stronger result that implies Gow's conjecture.

[☆] The research of the first author is supported by the Prometeo/Generalitat Valenciana, Proyectos MTM2013-40464-P. The second author gratefully acknowledges the support of the NSF (grant DMS-1201374) and the Simons Foundation Fellowship 305247.
[☆☆] The authors are grateful to the Mathematisches Forschungsinstitut Oberwolfach, where this project was started. Parts of the paper were written while the second author visited the Departments of Mathematics, University of Chicago. It is a pleasure to thank Ngo Bao Chau and the University of Chicago for generous hospitality and stimulating environment.
[☆☆☆] The authors are grateful to Roderick Gow and Martin Liebeck for helpful discussions, and to the referees for careful reading and helpful comments on the paper.

* Corresponding author.
E-mail addresses: gabriel.navarro@uv.es (G. Navarro), tiep@math.arizona.edu (P.H. Tiep).

http://dx.doi.org/10.1016/j.aim.2016.05.013
0001-8708/

1. Introduction

Reality questions on finite groups are almost as old as finite group theory itself, and they trace back to the work of Frobenius, Schur and Burnside.

An element g of a group G is **real** if g is conjugate to its inverse, and a character χ of G is **real-valued** if χ only takes real values. It is well-known that the number of real-valued irreducible complex characters of a finite group G is the number of conjugacy classes of G consisting of real elements, and a group G is **real** if all of its elements are real.

R. Gow's conjecture asserts that if G is a finite real group, then P/P' is an elementary abelian 2-group, where P is a Sylow 2-subgroup of G, and P' is the derived subgroup of P. This conjecture follows the main philosophy in finite group representation theory: global implies local or vice-versa.

In this paper we prove Gow's conjecture. The key is that we have found a more general statement with good inductive conditions which holds true (see Theorem 2.11 below). This statement has interest on its own, and in its simplest case already implies a much stronger version of Gow's conjecture.

Theorem A. *Let G be a finite group with a Sylow 2-subgroup P. If all the odd-degree irreducible characters of G are real-valued, then P/P' is elementary abelian.*

Our proof of Theorem A uses the Classification of Finite Simple Groups; in fact, we are able to reduce it to a problem on almost simple groups. The converse of Theorem A is not true, however, even for solvable groups.

There is no block version of Theorem A, except for the principal block. It is simply not true that if all the height zero characters in a 2-block B with defect group P are real-valued then P/P' is elementary abelian: D_{24}, for instance, is already a counterexample. On the other hand, we have the following non-trivial extension of Theorem A.

Theorem B. *Let G be a finite group with a Sylow 2-subgroup P. If all the odd-degree irreducible characters in the principal 2-block of G are real-valued, then P/P' is elementary abelian.*

It is not true that if G is a real group, then $P \in \mathrm{Syl}_2(G)$ is real (a statement that would imply Gow's conjecture). The smallest counterexample is the group $\mathsf{A}_4 : Q_8$.

Theorems A and B lead to an interesting problem: is it possible to give a group characterization of when all the odd-degree irreducible characters of G are real-valued? For solvable groups, we can show that this is the case if and only if $P = \mathbf{N}_G(P)$ and $P' = \Phi(P)$, the Frattini subgroup of P (i.e. the intersection of all maximal subgroups of P), see Theorem 2.13 below. But this is not true outside solvable groups, as $G = \mathsf{A}_5$ or $G = \mathsf{A}_9$ show us (in both directions). We also note that the converse of Theorem B is false, as shown by the example of $SL_3(2)$.

Finally, we have some evidence that there might be a version of Gow's conjecture for odd primes, but we will leave this problem for another place [25].