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A loop group method for affine harmonic maps into Lie groups

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We generalize the Uhlenbeck–Segal theory for harmonic maps into compact semi-simple Lie groups to general Lie groups equipped with torsion free bi-invariant connection.

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0. Introduction

Harmonic maps of Riemann surfaces into compact semi-simple Lie groups, equipped with a bi-invariant Riemannian metric, have been paid much attention by differential geometers as well as by mathematical physicists. In fact, harmonic maps of Riemann surfaces into compact Lie groups equipped with a bi-invariant Riemannian metric are called *principal chiral models* and intensively studied as toy models of gauge theory in mathematical physics [56].

Uhlenbeck established a fundamental theory of harmonic maps into the unitary group U_n, [55]. In particular she proved a factorization theorem for harmonic 2-spheres, the so called *uniton factorization*. Segal showed that harmonic maps into U_n are obtained by holomorphic curve into the based loop group of U_n [51]. Uhlenbeck–Segal theory actually works for any compact Lie groups [11,9]. Pedit, Wu and the first named author of the present paper generalized the Uhlenbeck–Segal theory to harmonic maps into compact Riemannian symmetric spaces, now referred to as the *generalized Weierstrass type representation* [19].

It turned out that the compactness of the target space is not necessary, as long as one considers only surfaces away from singularities and considers groups with bi-invariant metric, Riemannian or pseudo-Riemannian. From a global point of view the construction principle will generally produce surfaces with singularities. A typical example for this are the spacelike CMC surfaces in Minkowski 3-space. In this case one considers harmonic maps into the Riemannian symmetric space $\mathbb{H}^2 = \mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2$. Harmonic maps into $\mathbb{H}^2$ are closely related to the so-called tt*-geometry [20]. It is a very important and difficult problem to find (and describe) globally smooth solutions for non-compact target spaces.

If one wants to generalize the loop group approach for harmonic maps into symmetric spaces to general homogeneous spaces, where the Lie group has only a left-invariant metric, one encounters a completely new situation. Clearly, the case of harmonic maps into Lie groups is a first interesting case. Since abelian groups always have bi-invariant metrics, the next interesting case is the one of 3-dimensional Lie groups (with left-invariant metric).

As a matter of fact, and largely independent of the issues discussed above, during the last ten years or so, minimal surfaces into 3-dimensional Lie groups have been studied extensively [44]. In particular, minimal surfaces in 3-dimensional Lie groups, equipped with a left-invariant metric, have been investigated intensively. With the exception of the space $\mathbb{S}^2 \times \mathbb{R}$, the other seven model spaces of Thurston geometries obviously have or can be given the structure of a Lie group. These are the following spaces; the Euclidean 3-space $\mathbb{E}^3$, the unit 3-sphere $\mathbb{S}^3$, the hyperbolic 3-space $\mathbb{H}^3$, the model space Nil_3 of nilgeometry, the universal covering group $\widetilde{\mathrm{SL}_2\mathbb{R}}$, the space Sol_3 of solvgeometry and the product space $\mathbb{H}^2 \times \mathbb{R}$ [53]. The metrics on these groups are generally only left-invariant with respect to the Lie group structure. Only the Euclidean 3-space $\mathbb{E}^3$ and the 3-sphere $\mathbb{S}^3$ admit bi-invariant Riemannian metrics.