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Duality and (q-)multiple zeta values

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ABSTRACT

Following Bachmann's recent work on bi-brackets and multiple Eisenstein series, Zudilin introduced the notion of multiple q-zeta brackets, which provides a q-analog of multiple zeta values possessing both shuffle as well as quasi-shuffle relations. The corresponding products are related in terms of duality. In this work we study Zudilin's duality construction in the context of classical multiple zeta values as well as various q-analogs of multiple zeta values. Regarding the former we identify the derivation relation of order two with a Hoffman–Ohno type relation. Then we describe relations between the Ohno–Okuda–Zudilin q-multiple zeta values and the Schlesinger–Zudilin q-multiple zeta values.

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Infinitesimal bialgebra
Rota–Baxter algebra

1. Introduction

Multiple zeta values (MZVs) are nested sums of depth $n \in \mathbb{N}$ and weight $\sum_{i=1}^{n} k_i > n$, for positive integers $k_1 > 1$, $k_i > 0$, $i = 2, \ldots, n$

$$\zeta(k_1, \ldots, k_n) := \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}. \tag{1}$$

They arise in various contexts, e.g., number theory, algebraic geometry, algebra, as well as knot theory. The origins of the modern systematic treatment of MZVs can be traced back to [7,26,8]. Moreover, MZVs and their generalisations, i.e., multiple polylogarithms, play an important role in quantum field theory [3].

It is well-known that the nested sums in (1) can also be written as iterated Chen integrals, which induces *shuffle relations* among MZVs thanks to integration by parts. Multiplying MZVs directly, on the other hand, yields *quasi-shuffle relations*. Comparison of the different products results in a large collection of so-called *double shuffle relations* among MZVs [11,29,9,25]. Amending a regularization procedure with respect to the formally defined single zeta value $t := \zeta(1)$ leads to *regularized double shuffle relations*, which conjecturally give all linear relations among MZVs [12].

A particular change of variables in the context of the integral representation of MZVs gives way to a peculiar set of relations subsumed under the notion of *duality* for MZVs. As an example we state the identity

$$\zeta(5, 1) = \zeta(3, 1, 1, 1).$$

A precise understanding of the mathematical relation between the notion of duality and the aforementioned double shuffle structure is part of a class of important open problems in the theory of MZVs. See [12] for details.

Generalizations of the real-valued nested sums in (1) to power series in $\mathbb{Q}[[q]]$ are commonly known as q-analogues of MZVs. A particular example of such q-MZVs is due to Bradley and Zhao [2,27], who extended a q-analog of the Riemann zeta function introduced by Kaneko et al. [15]. More recently, several q-analogues of MZVs were shown to satisfy – regularized – double shuffle relations [5,22,21,23]. The notion of duality in the context of those q-MZVs is more complicated. See [28] for details. Inspired by Bachmann's intriguing work [1], Zudilin presents in [30] a particular model called *multiple q-zeta brackets*, which possesses a natural quasi-shuffle product. After multiplying Zudilin's multiple q-zeta brackets with a certain positive integer power of $1-q$ one obtains ordinary MZVs in the classical limit $q \to 1$. The key result in [30] is an algebraic duality-type construction that permits to deduce a shuffle product for multiple q-zeta brackets from