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Matrix positivity preservers in fixed dimension. I

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A B S T R A C T

A classical theorem proved in 1942 by I.J. Schoenberg describes all real-valued functions that preserve positivity when applied entrywise to positive semidefinite matrices of arbitrary size; such functions are necessarily analytic with non-negative Taylor coefficients. Despite the great deal of interest generated by this theorem, a characterization of functions preserving positivity for matrices of fixed dimension is not known.

In this paper, we provide a complete description of polynomials of degree N that preserve positivity when applied entrywise to matrices of dimension N. This is the key step for us then to obtain negative lower bounds on the coefficients of analytic functions so that these functions preserve positivity in a prescribed dimension. The proof of the main technical inequality is representation theoretic, and employs the theory of Schur polynomials. Interpreted in the context of linear pencils of matrices, our main results provide a closed-form expression for the lowest critical value, revealing at the same time an unexpected spectral discontinuity phenomenon.

Tight linear matrix inequalities for Hadamard powers of matrices and a sharp asymptotic bound for the matrix-cube problem involving Hadamard powers are obtained as applications. Positivity preservers are also naturally interpreted as solutions of a variational inequality involving generalized Rayleigh quo-

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tients. This optimization approach leads to a novel description of the simultaneous kernels of Hadamard powers, and a family of stratifications of the cone of positive semidefinite matrices.

Contents

1. Introduction and main results

Transformations, linear or not, which preserve matrix structures with positivity constraints have been recently studied in at least three distinct frameworks: statistical mechanics and the geometry of polynomials [8–10]; global optimization algorithms based on the cone of hyperbolic or positive definite polynomials [5,24,39]; the statistics of big data, having the correlation matrix of a large number of random variables as the central object [4,25,32,40,46]. The present article belongs in the latter two categories, although the main result may be of independent algebraic interest.

To describe the contents of this paper, we adopt some terminology. For a set $K \subset \mathbb{C}$ and an integer $N \geq 1$, denote by $\mathcal{P}_N(K)$ the cone of positive semidefinite $N \times N$ matrices with entries in K. A function $f : K \to \mathbb{C}$ naturally acts entrywise on $\mathcal{P}_N(K)$, so that $f[A] := (f(a_{ij}))$ for any $A = (a_{ij}) \in \mathcal{P}_N(K)$. Akin to the theory of positive definite functions, it is natural to seek characterizations of those functions f such that $f[A]$ is positive semidefinite for all $A \in \mathcal{P}_N(K)$. A well-known theorem of Schoenberg [42] states that $f[A]$ is positive semidefinite for all $A \in \mathcal{P}_N([-1, 1])$ of all dimensions $N \geq 1$ if and only if f is absolutely monotonic on $[0, 1]$ (i.e., analytic with non-negative Taylor coefficients). To put Schoenberg's 1942 article in historical perspective, we have to recall that the theory of absolute monotone functions was already established by S. Bernstein [3]. Also, it is worth mentioning that Schoenberg was working around that time on the related and more general question of isometrically embedding positive definite metrics into Hilbert space; see, for instance, [45]. The parallel theory of matrix monotone functions,