Advances in Mathematics 298 (2016) 413–447

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Motivic infinite cyclic covers

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ARTICLE INFO

Article history:
Received 23 December 2014
Received in revised form 8 April 2016
Accepted 25 April 2016
Available online 10 May 2016
Communicated by Tony Pantev

Keywords:
Infinite cyclic cover of finite type
Motivic infinite cyclic cover
Milnor fiber
Motivic Milnor fiber
Betti realization

ABSTRACT

We associate with an infinite cyclic cover of a punctured neighborhood of a simple normal crossing divisor on a complex quasi-projective manifold (under certain finiteness conditions) an element in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$, which we call *motivic infinite cyclic cover*, and show its birational invariance. Our construction provides a unifying approach for the Denef–Loeser motivic Milnor fiber of a complex hypersurface singularity germ, and the motivic Milnor fiber of a rational function, respectively.

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http://dx.doi.org/10.1016/j.aim.2016.04.019
0001-8708/

1. Introduction

Infinite cyclic covers are fundamental objects of study in topology (e.g., in knot theory [28], but see also [25]) and algebraic geometry (e.g., for the study of Alexander-type invariants of complex hypersurface complements, see [9,10,16,17,23]).

The Milnor fiber of a hypersurface singularity germ (cf. [24]), can be viewed as an example of an infinite cyclic cover, since it is a retract of the infinite cyclic cover of the complement to the germ in a small ball about the singular point. Moreover, in this interpretation, the monodromy of the Milnor fiber corresponds to the action of the generator of the group of deck transformations of the infinite cyclic cover (cf. Section 2 below; but see also [18], where such an identification was used to define an abelian version of the Milnor fiber, and [8] for a detailed discussion in the homogeneous case).

Motivated by connections between the Igusa zeta functions, Bernstein–Sato polynomials and the topology of hypersurface singularities, Denef and Loeser defined in [5–7] the motivic zeta function and the motivic Milnor fiber of a hypersurface singularity germ; the latter is a virtual variety endowed with an action of the group scheme of roots of unity, from which one can retrieve several invariants of the (topological) Milnor fiber, e.g., the Hodge–Steenbrink spectrum, Euler characteristic, etc. The motivic Milnor fiber has also appeared in the Soibelman–Kontsevich theory of motivic Donaldson–Thomas invariants.

In this paper, we attach to an infinite cyclic cover associated to a punctured neighborhood of a simple normal crossing divisor E on a complex quasi-projective manifold X, an element in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$ of algebraic $\mathbb{C}$-varieties endowed with a good action of the pro-finite group $\hat{\mu} = \lim \mu_n$ of roots of unity, which we call a *motivic infinite cyclic cover*; see Section 3 for details. (Our terminology is inspired by the standard notion of "motivic Milnor fiber", cf. [7].) Among other consequences, this construction allows us to define a motivic infinite cyclic cover of a hypersurface singularity germ complement, which as we show later on coincides (in the localization of $K_0(\mathrm{Var}_{\mathbb{C}}^{\hat{\mu}})$ at the class $\mathbb{L}$ of the affine line) with the above-mentioned Denef–Loeser motivic Milnor fiber. Our class of coverings guarantees certain finiteness conditions (see Definition 2.1) which are present in the case of Milnor fibers, but which are not satisfied in general. Note that while these infinite cyclic covers are complex manifolds, they are not algebraic varieties in general. This paper provides an algebro-geometric interpretation of such covering spaces.

Our construction of motivic infinite cyclic covers is topological in the sense that it does not make use of arc spaces as is the case in earlier constructions of motivic Milnor fibers. We rely instead on the weak factorization theorem [1,4]. One of our main results, Theorem 3.7, shows that our notion of motivic infinite cyclic cover is a birational invariant, or equivalently, it is an invariant of the punctured neighborhood of E in X. Moreover, in Section 4 we show that the Betti realization of the motivic infinite cyclic cover is given by the cohomology with compact support of the infinite cyclic cover of the punctured neighborhood, e.g., their Euler characteristics coincide.